

# EXTENDED T-SYSTEM OF TYPE $G_2$

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**ABSTRACT.** We prove a family of 3-term relations in the Grothendieck ring of the category of finite-dimensional modules over the affine quantum algebra of type  $G_2$  extending the celebrated  $T$ -system relations of type  $G_2$ . We show that these relations can be used to compute classes of certain irreducible modules, including classes of all minimal affinizations of type  $G_2$ . We use this result to obtain explicit formulas for dimensions of all participating modules.

## 1. INTRODUCTION

Kirillov-Reshetikhin modules are simplest examples of irreducible finite-dimensional modules over quantum affine algebras, and the  $T$ -system is a famous family of short exact sequences of tensor products of Kirillov-Reshetikhin modules, see [KR90], [KNS94], [Nak03], [Her06]. There are numerous applications of the  $T$ -systems in representation theory, combinatorics and integrable systems, see the survey [KNS11].

Minimal affinizations of quantum affine algebras form an important family of irreducible modules which contains the Kirillov-Reshetikhin modules, see [CP95b]. A procedure to extend the  $T$ -system to a larger set of relations to include the minimal affinization was described in [MY11b], where it was conjectured to work in all types. In [MY11b] this procedure was carried out in types  $A$  and  $B$ . In this paper, we show the existence of the extended  $T$ -system for type  $G_2$ .

We work with the quantum affine algebra  $U_q\hat{\mathfrak{g}}$  of type  $G_2$ . The irreducible finite-dimensional modules of quantum affine algebras are parameterized by the highest  $l$ -weights or Drinfeld polynomials. Let  $\mathcal{T}$  be an irreducible  $U_q\hat{\mathfrak{g}}$ -module such that zeros of all Drinfeld polynomials belong to a lattice  $aq^{\mathbb{Z}}$  for some  $a \in \mathbb{C}^\times$ . Following [MY11b], we define the left, right, and bottom modules, denoted by  $\mathcal{L}$ ,  $\mathcal{R}$ ,  $\mathcal{B}$  respectively. The Drinfeld polynomials of left, right, and bottom modules are obtained by stripping the rightmost, leftmost, and both left- and rightmost zeros of the union of zeros of the Drinfeld polynomials of the top module  $\mathcal{T}$ .

Then the relations of the extended  $T$ -system have the form  $[\mathcal{L}][\mathcal{R}] = [\mathcal{T}][\mathcal{B}] + [\mathcal{S}]$ , where  $[\cdot]$  denotes the equivalence class of a  $U_q\hat{\mathfrak{g}}$ -module in the Grothendieck ring of the category of finite-dimensional representations of  $U_q\hat{\mathfrak{g}}$ . Moreover, in all cases the modules  $\mathcal{T} \otimes \mathcal{B}$  and  $\mathcal{S}$  are irreducible.

We start with minimal affinizations as the top modules  $\mathcal{T}$ , then the left, right and bottom modules are minimal affinizations as well. We compute  $\mathcal{S}$  and decompose it as a product of irreducible modules which we call sources. It turns out that the sources are not always minimal affinizations. Therefore, we follow up with taking the sources as top modules and compute new left, right, bottom modules, and sources. Then we use all new modules obtained on a previous step as top modules and so on.

We end up with several families of modules which we denote by  $\mathcal{B}_{k,\ell}^{(s)}$ ,  $\mathcal{C}_{k,\ell}^{(s)}$ ,  $\mathcal{D}_{k,\ell}^{(s)}$ ,  $\mathcal{E}_{k,\ell}^{(s)}$ ,  $\mathcal{F}_{k,\ell}^{(s)}$ ,  $\tilde{\mathcal{B}}_{k,\ell}^{(s)}$ ,  $\tilde{\mathcal{C}}_{k,\ell}^{(s)}$ ,  $\tilde{\mathcal{D}}_{k,\ell}^{(s)}$ ,  $\tilde{\mathcal{E}}_{k,\ell}^{(s)}$ ,  $\tilde{\mathcal{F}}_{k,\ell}^{(s)}$ , where  $s \in \mathbb{Z}$ ,  $k, \ell \in \mathbb{Z}_{\geq 0}$ . This is the minimal set of modules which contains all minimal affinizations (these are modules  $\mathcal{B}_{k,\ell}^{(s)}$ ,  $\tilde{\mathcal{B}}_{k,\ell}^{(s)}$ ) and which is closed under our set of relations. Namely, if any of the above modules is chosen as a top module then the left, right, bottom modules and all sources belong to this set as well, see Theorems 3.4, 7.4.

We show that the extended T-system allows us to compute the modules  $\mathcal{B}_{k,\ell}^{(s)}$ ,  $\mathcal{C}_{k,\ell}^{(s)}$ ,  $\mathcal{D}_{k,\ell}^{(s)}$ ,  $\mathcal{E}_{k,\ell}^{(s)}$ ,  $\mathcal{F}_{k,\ell}^{(s)}$  recursively in terms of fundamental modules, see Proposition 3.6. We use this to compute the dimensions of all participating modules, in particular, we give explicit formulas for dimensions of all minimal affinizations of type  $G_2$ , see Theorem 8.1. We hope further, that one can use the extended  $T$ -system to obtain the decomposition of all participating modules as the  $U_q\mathfrak{g}$ -modules.

Let us point out some similarities and differences with types  $A$  and  $B$ . The type  $A$ , the extended T-system is closed within the class of minimal affinizations, meaning that all sources are minimal affinizations as well. In type  $B$ , the extended T-system is not closed within the class of minimal affinizations, but it is closed in the class of so called snake modules, see [MY11b]. For the proofs and computations it is important that all modules participating in extended T-systems of types  $A$  and  $B$  are thin and special, moreover their  $q$ -characters are known explicitly in terms of skew Young tableaux in type  $A$ , and in terms of path models in type  $B$ , see [Che87], [NT98], [MY11a], [MY11b].

In general the modules of the extended  $T$ -system of type  $G_2$  are not thin and at the moment there is no combinatorial description of their  $q$ -characters. However, all modules turn out to be either special or anti-special. Therefore we are able to use the FM algorithm, see [FM01], to compute the sufficient information about  $q$ -characters in order to complete the proofs. Note, that a priori it is not obvious that the extended T-system will be closed within special or anti-special modules. Moreover, since the  $q$ -characters of  $G_2$  modules are not known explicitly, the property of being special or anti-special had to be established in each case, see Theorems 3.3, 7.2.

Note that in general the minimal affinizations of types  $C, D, E, F$  are neither special nor anti-special, therefore the methods of this paper cannot be applied in those cases.

There is a remarkable conjecture on the cluster algebra relations in the category of finite-dimensional representations of quantum affine algebras of type  $A, D, E$ , see [HL10]. Taking into account the work of [IIKKN10a], [IIKKN10b], one could expect that the conjecture of [HL10] can be formulated for other types as well, in particular for type  $G_2$ . We expect that the extended  $T$ -system is a part of cluster algebra relations.

The paper is organized as follows. In Section 2, we give some background material. In Section 3, we define the modules  $\mathcal{B}_{k,\ell}^{(s)}$ ,  $\mathcal{C}_{k,\ell}^{(s)}$ ,  $\mathcal{D}_{k,\ell}^{(s)}$ ,  $\mathcal{E}_{k,\ell}^{(s)}$ ,  $\mathcal{F}_{k,\ell}^{(s)}$  and state our main result, Theorem 3.4. In Section 4, we prove that the modules  $\mathcal{B}_{k,\ell}^{(s)}$ ,  $\mathcal{C}_{k,\ell}^{(s)}$ ,  $\mathcal{D}_{k,\ell}^{(s)}$ ,  $\mathcal{E}_{k,\ell}^{(s)}$ ,  $\mathcal{F}_{k,\ell}^{(s)}$  are special. In Section 5, we prove Theorem 3.4. In Section 6, we prove that the module  $\mathcal{T} \otimes \mathcal{B}$  is irreducible for each relation in the extended T-system. In Section 7, we deduce the extended T-system for the modules  $\tilde{\mathcal{B}}_{k,\ell}^{(s)}$ ,  $\tilde{\mathcal{C}}_{k,\ell}^{(s)}$ ,  $\tilde{\mathcal{D}}_{k,\ell}^{(s)}$ ,  $\tilde{\mathcal{E}}_{k,\ell}^{(s)}$ ,  $\tilde{\mathcal{F}}_{k,\ell}^{(s)}$ . In Section 8, we compute the dimensions of the modules in the extended T-systems.

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## 2. BACKGROUND

**2.1. Cartan data.** Let  $\mathfrak{g}$  be a complex simple Lie algebra of type  $G_2$  and  $\mathfrak{h}$  a Cartan subalgebra of  $\mathfrak{g}$ . Let  $I = \{1, 2\}$ . We choose simple roots  $\alpha_1, \alpha_2$  and scalar product  $(\cdot, \cdot)$  such that

$$(\alpha_1, \alpha_1) = 2, (\alpha_1, \alpha_2) = -3, (\alpha_2, \alpha_2) = 6.$$

Let  $\{\alpha_1^\vee, \alpha_2^\vee\}$  and  $\{\omega_1, \omega_2\}$  be the sets of simple coroots and fundamental weights respectively. Let  $C = (C_{ij})_{i,j \in I}$  denote the Cartan matrix, where  $C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$ . Let  $r_1 = 1, r_2 = 3$ ,  $D = \text{diag}(r_1, r_2)$  and  $B = DC$ . Then

$$C = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}.$$

Let  $Q$  (resp.  $Q^+$ ) and  $P$  (resp.  $P^+$ ) denote the  $\mathbb{Z}$ -span (resp.  $\mathbb{Z}_{\geq 0}$ -span) of the simple roots and fundamental weights respectively. Let  $\leq$  be the partial order on  $P$  in which  $\lambda \leq \lambda'$  if and only if  $\lambda' - \lambda \in Q^+$ .

Let  $\hat{\mathfrak{g}}$  denote the untwisted affine algebra corresponding to  $\mathfrak{g}$ . Fix a  $q \in \mathbb{C}^\times$ , not a root of unity. Let  $q_i = q^{r_i}, i = 1, 2$ . Define the  $q$ -numbers,  $q$ -factorial and  $q$ -binomial:

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! := [n]_q [n-1]_q \cdots [1]_q, \quad \begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{[n]_q!}{[n-m]_q! [m]_q!}.$$

**2.2. Quantum affine algebra.** The quantum affine algebra  $U_q \hat{\mathfrak{g}}$  in Drinfeld's new realization, see [Dri88], is generated by  $x_{i,n}^\pm$  ( $i \in I, n \in \mathbb{Z}$ ),  $k_i^{\pm 1}$  ( $i \in I$ ),  $h_{i,n}$  ( $i \in I, n \in \mathbb{Z} \setminus \{0\}$ ) and central elements  $c^{\pm 1/2}$ , subject to the following relations:

$$\begin{aligned} k_i k_j &= k_j k_i, \quad k_i h_{j,n} = h_{j,n} k_i, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1, \\ k_i x_{j,n}^\pm k_i^{-1} &= q^{\pm B_{ij}} x_{j,n}^\pm, \\ [h_{i,n}, x_{j,m}^\pm] &= \pm \frac{1}{n} [n B_{ij}]_q c^{\mp |n|/2} x_{j,n+m}^\pm, \\ x_{i,n+1}^\pm x_{j,m}^\pm - q^{\pm B_{ij}} x_{j,m}^\pm x_{i,n+1}^\pm &= q^{\pm B_{ij}} x_{i,n}^\pm x_{j,m+1}^\pm - x_{j,m+1}^\pm x_{i,n}^\pm, \\ [h_{i,n}, h_{j,m}] &= \delta_{n,-m} \frac{1}{n} [n B_{ij}]_q \frac{c^n - c^{-n}}{q - q^{-1}}, \\ [x_{i,n}^+, x_{j,m}^-] &= \delta_{ij} \frac{c^{(n-m)/2} \phi_{i,n+m}^+ - c^{-(n-m)/2} \phi_{i,n+m}^-}{q_i - q_i^{-1}}, \\ \sum_{\pi \in \Sigma_s} \sum_{k=0}^s (-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_{q_i} x_{i,n_{\pi(1)}}^\pm \cdots x_{i,n_{\pi(k)}}^\pm x_{j,m}^\pm x_{i,n_{\pi(k+1)}}^\pm \cdots x_{i,n_{\pi(s)}}^\pm &= 0, \quad s = 1 - C_{ij}, \end{aligned}$$

for all sequences of integers  $n_1, \dots, n_s$ , and  $i \neq j$ , where  $\Sigma_s$  is the symmetric groups on  $s$  letters and  $\phi_{i,n}^\pm$ 's are determined by the formula

$$\phi_i^\pm(u) := \sum_{n=0}^{\infty} \phi_{i,\pm n}^\pm u^{\pm n} = k_i^{\pm 1} \exp \left( \pm (q - q^{-1}) \sum_{m=1}^{\infty} h_{i,\pm m} u^{\pm m} \right). \quad (2.1)$$

There exist a coproduct, counit and antipode making  $U_q \hat{\mathfrak{g}}$  into a Hopf algebra.

The quantum affine algebra  $U_q \hat{\mathfrak{g}}$  contains two standard quantum affine algebras of type  $A_1$ . The first one is  $U_{q_1}(\hat{\mathfrak{sl}}_2)$  generated by  $x_{1,n}^\pm$  ( $n \in \mathbb{Z}$ ),  $k_1^{\pm 1}$ ,  $h_{1,n}$  ( $n \in \mathbb{Z} \setminus \{0\}$ ) and central elements  $c^{\pm 1/2}$ . The second one is  $U_{q_2}(\hat{\mathfrak{sl}}_2)$  generated by  $x_{2,n}^\pm$  ( $n \in \mathbb{Z}$ ),  $k_2^{\pm 1}$ ,  $h_{2,n}$  ( $n \in \mathbb{Z} \setminus \{0\}$ ) and central elements  $c^{\pm 1/2}$ .

The subalgebra of  $U_q \hat{\mathfrak{g}}$  generated by  $(k_i^\pm)_{i \in I}$ ,  $(x_{i,0}^\pm)_{i \in I}$  is a Hopf subalgebra of  $U_q \hat{\mathfrak{g}}$  and is isomorphic as a Hopf algebra to  $U_q \mathfrak{g}$ , the quantized enveloping algebra of  $\mathfrak{g}$ . In this way,  $U_q \hat{\mathfrak{g}}$ -modules restrict to  $U_q \mathfrak{g}$ -modules.

**2.3. Finite-dimensional representations and  $q$ -characters.** In this section, we recall the standard facts about finite-dimensional representations of  $U_q \hat{\mathfrak{g}}$  and  $q$ -characters of these representations, see [CP94], [CP95a], [FR98], [MY11b].

A representation  $V$  of  $U_q \hat{\mathfrak{g}}$  is of type 1 if  $c^{\pm 1/2}$  acts as the identity on  $V$  and

$$V = \bigoplus_{\lambda \in P} V_\lambda, \quad V_\lambda = \{v \in V : k_i v = q^{(\alpha_i, \lambda)} v\}. \quad (2.2)$$

In the following, all representations will be assumed to be finite-dimensional and of type 1 without further comment. The decomposition (2.2) of a finite-dimensional representation  $V$  into its  $U_q \mathfrak{g}$ -weight spaces can be refined by decomposing it into the Jordan subspaces of the mutually commuting operators  $\phi_{i,\pm r}^\pm$ , see [FR98]:

$$V = \bigoplus_{\gamma} V_\gamma, \quad \gamma = (\gamma_{i,\pm r}^\pm)_{i \in I, r \in \mathbb{Z}_{\geq 0}}, \quad \gamma_{i,\pm r}^\pm \in \mathbb{C}, \quad (2.3)$$

where

$$V_\gamma = \{v \in V : \exists k \in \mathbb{N}, \forall i \in I, m \geq 0, (\phi_{i,\pm m}^\pm - \gamma_{i,\pm m}^\pm)^k v = 0\}.$$

If  $\dim(V_\gamma) > 0$ , then  $\gamma$  is called an  $l$ -weight of  $V$ . For every finite dimensional representation of  $U_q \hat{\mathfrak{g}}$ , the  $l$ -weights are known, see [FR98], to be of the form

$$\gamma_i^\pm(u) := \sum_{r=0}^{\infty} \gamma_{i,\pm r}^\pm u^{\pm r} = q_i^{\deg Q_i - \deg R_i} \frac{Q_i(uq_i^{-1})R_i(uq_i)}{Q_i(uq_i)R_i(uq_i^{-1})}, \quad (2.4)$$

where the right hand side is to be treated as a formal series in positive (resp. negative) integer powers of  $u$ , and  $Q_i, R_i$  are polynomials of the form

$$Q_i(u) = \prod_{a \in \mathbb{C}^\times} (1 - ua)^{w_{i,a}}, \quad R_i(u) = \prod_{a \in \mathbb{C}^\times} (1 - ua)^{x_{i,a}}, \quad (2.5)$$

for some  $w_{i,a}, x_{i,a} \in \mathbb{Z}_{\geq 0}, i \in I, a \in \mathbb{C}^\times$ . Let  $\mathcal{P}$  denote the free abelian multiplicative group of monomials in infinitely many formal variables  $(Y_{i,a})_{i \in I, a \in \mathbb{C}^\times}$ . There is a bijection  $\gamma$  from  $\mathcal{P}$  to the set of  $l$ -weights of finite-dimensional modules such that for the monomial  $m = \prod_{i \in I, a \in \mathbb{C}^\times} Y_{i,a}^{w_{i,a} - x_{i,a}}$ , the  $l$ -weight  $\gamma(m)$  is given by (2.4), (2.5).

Let  $\mathbb{Z}\mathcal{P} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$  be the group ring of  $\mathcal{P}$ . For  $\chi \in \mathbb{Z}\mathcal{P}$ , we write  $m \in \mathcal{P}$  if the coefficient of  $m$  in  $\chi$  is non-zero.

The  $q$ -character of a  $U_q\hat{\mathfrak{g}}$ -module  $V$  is given by

$$\chi_q(V) = \sum_{m \in \mathcal{P}} \dim(V_m) m \in \mathbb{Z}\mathcal{P},$$

where  $V_m = V_{\gamma(m)}$ .

Let  $\text{Rep}(U_q\hat{\mathfrak{g}})$  be the Grothendieck ring of finite-dimensional representations of  $U_q\hat{\mathfrak{g}}$  and  $[V] \in \text{Rep}(U_q\hat{\mathfrak{g}})$  the class of a finite-dimensional  $U_q\hat{\mathfrak{g}}$ -module  $V$ . The  $q$ -character map defines an injective ring homomorphism, see [FR98],

$$\chi_q : \text{Rep}(U_q\hat{\mathfrak{g}}) \rightarrow \mathbb{Z}\mathcal{P}.$$

For any finite-dimensional representation  $V$  of  $U_q\hat{\mathfrak{g}}$ , denote by  $\mathcal{M}(V)$  the set of all monomials in  $\chi_q(V)$ . For each  $j \in I$ , a monomial  $m = \prod_{i \in I, a \in \mathbb{C}^\times} Y_{i,a}^{u_{i,a}}$ , where  $u_{i,a}$  are some integers, is said to be  $j$ -dominant (resp.  $j$ -anti-dominant) if and only if  $u_{j,a} \geq 0$  (resp.  $u_{j,a} \leq 0$ ) for all  $a \in \mathbb{C}^\times$ . A monomial is called *dominant* (resp. *anti-dominant*) if and only if it is  $j$ -dominant (resp.  $j$ -anti-dominant) for all  $j \in I$ . Let  $\mathcal{P}^+ \subset \mathcal{P}$  denote the set of all dominant monomials.

Let  $V$  be a representation of  $U_q\hat{\mathfrak{g}}$  and  $m \in \mathcal{M}(V)$  a monomial. A non-zero vector  $v \in V_m$  is called a *highest  $l$ -weight vector* with *highest  $l$ -weight*  $\gamma(m)$  if

$$x_{i,r}^+ \cdot v = 0, \quad \phi_{i,\pm t}^\pm \cdot v = \gamma(m)_{i,\pm t}^\pm v, \quad \forall i \in I, r \in \mathbb{Z}, t \in \mathbb{Z}_{\geq 0}.$$

The module  $V$  is called a *highest  $l$ -weight representation* if  $V = U_q\hat{\mathfrak{g}} \cdot v$  for some highest  $l$ -weight vector  $v \in V$ .

It is known, see [CP94], [CP95a], that for each  $m_+ \in \mathcal{P}^+$  there is a unique finite-dimensional irreducible representation, denoted  $L(m_+)$ , of  $U_q\hat{\mathfrak{g}}$  that is highest  $l$ -weight with highest  $l$ -weight  $\gamma(m_+)$ , and moreover every finite-dimensional irreducible  $U_q\hat{\mathfrak{g}}$ -module is of this form for some  $m_+ \in \mathcal{P}^+$ . Also, if  $m_+, m'_+ \in \mathcal{P}^+$  and  $m_+ \neq m'_+$ , then  $L(m_+) \not\cong L(m'_+)$ . For  $m_+ \in \mathcal{P}^+$ , we use  $\chi_q(m_+)$  to denote  $\chi_q(L(m_+))$ .

The following lemma is well-known.

**Lemma 2.1.** *Let  $m_1, m_2$  be two monomials. Then  $L(m_1 m_2)$  is a sub-quotient of  $L(m_1) \otimes L(m_2)$ . In particular,  $\mathcal{M}(L(m_1 m_2)) \subseteq \mathcal{M}(L(m_1)) \mathcal{M}(L(m_2))$ .  $\square$*

For  $b \in \mathbb{C}^\times$ , define the shift of spectral parameter map  $\tau_b : \mathbb{Z}\mathcal{P} \rightarrow \mathbb{Z}\mathcal{P}$  to be a homomorphism of rings sending  $Y_{i,a}^{\pm 1}$  to  $Y_{i,ab}^{\pm 1}$ . Let  $m_1, m_2 \in \mathcal{P}^+$ . If  $\tau_b(m_1) = m_2$ , then

$$\tau_b \chi_q(m_1) = \chi_q(m_2). \quad (2.6)$$

A finite-dimensional  $U_q\hat{\mathfrak{g}}$ -module  $V$  is said to be *special* if and only if  $\mathcal{M}(V)$  contains exactly one dominant monomial. It is called *anti-special* if and only if  $\mathcal{M}(V)$  contains exactly one anti-dominant monomial. It is called *thin* if and only if no  $l$ -weight space of  $V$  has dimension greater than 1. It is said to be *prime* if and only if it is not isomorphic to a tensor product of two non-trivial  $U_q\hat{\mathfrak{g}}$ -modules, see [CP97]. Clearly, if a module is special or anti-special, then it is irreducible.

Define  $A_{i,a} \in \mathcal{P}$ ,  $i \in I, a \in \mathbb{C}^\times$ , by

$$A_{1,a} = Y_{1,aq} Y_{1,aq^{-1}} Y_{2,a}^{-1}, \quad A_{2,a} = Y_{2,aq^3} Y_{2,aq^{-3}} Y_{1,aq^{-2}}^{-1} Y_{1,a}^{-1} Y_{1,aq^2}^{-1}.$$

Let  $\mathcal{Q}$  be the subgroup of  $\mathcal{P}$  generated by  $A_{i,a}, i \in I, a \in \mathbb{C}^\times$ . Let  $\mathcal{Q}^\pm$  be the monoids generated by  $A_{i,a}^{\pm 1}, i \in I, a \in \mathbb{C}^\times$ . There is a partial order  $\leq$  on  $\mathcal{P}$  in which

$$m \leq m' \text{ if and only if } m'm^{-1} \in \mathcal{Q}^+. \quad (2.7)$$

For all  $m_+ \in \mathcal{P}^+$ ,  $\mathcal{M}(L(m_+)) \subset m_+ \mathcal{Q}^-$ , see [FM01].

A monomial  $m$  is called *right negative* if and only if  $\forall a \in \mathbb{C}^\times$  and  $\forall i \in I$ , we have the following property: if the power of  $Y_{i,a}$  is non-zero and the power of  $Y_{j,aq^k}$  is zero for all  $j \in I, k \in \mathbb{Z}_{>0}$ , then the power of  $Y_{i,a}$  is negative. For  $i \in I, a \in \mathbb{C}^\times$ ,  $A_{i,a}^{-1}$  is right-negative. A product of right-negative monomials is right-negative. If  $m$  is right-negative, then  $m' \leq m$  implies that  $m'$  is right-negative.

**2.4. Minimal affinizations of  $U_q \mathfrak{g}$ -modules.** Let  $\lambda = k\omega_1 + \ell\omega_2$ . A simple  $U_q \hat{\mathfrak{g}}$ -module  $L(m_+)$  is called a *minimal affinization* of  $V(\lambda)$  if and only if  $m_+$  is one of the following monomials

$$\left( \prod_{i=0}^{\ell-1} Y_{2,aq^{6i}} \right) \left( \prod_{i=0}^{k-1} Y_{1,aq^{6\ell+2i+1}} \right), \quad \left( \prod_{i=0}^{k-1} Y_{1,aq^{2i}} \right) \left( \prod_{i=0}^{\ell-1} Y_{2,aq^{2k+6i+5}} \right),$$

for some  $a \in \mathbb{C}^\times$ , see [CP95b]. In particular, when  $k = 0$  or  $\ell = 0$ , the minimal affinization  $L(m_+)$  is called a *Kirillov-Reshetikhin module*.

Let  $L(m_+)$  be a Kirillov-Reshetikhin module. It is shown in [Her06] that any non-highest monomial in  $\mathcal{M}(L(m_+))$  is right-negative and in particular  $L(m_+)$  is special.

**2.5.  $q$ -characters of  $U_q \hat{\mathfrak{sl}}_2$ -modules and the FM algorithm.** The  $q$ -characters of  $U_q \hat{\mathfrak{sl}}_2$ -modules are well-understood, see [CP91], [FR98]. We recall the results here.

Let  $W_k^{(a)}$  be the irreducible representation  $U_q \hat{\mathfrak{sl}}_2$  with highest weight monomial

$$X_k^{(a)} = \prod_{i=0}^{k-1} Y_{aq^{k-2i-1}},$$

where  $Y_a = Y_{1,a}$ . Then the  $q$ -character of  $W_k^{(a)}$  is given by

$$\chi_q(W_k^{(a)}) = X_k^{(a)} \sum_{i=0}^k \prod_{j=0}^{i-1} A_{aq^{k-2j}}^{-1},$$

where  $A_a = Y_{aq^{-1}} Y_{aq}$ .

For  $a \in \mathbb{C}^\times, k \in \mathbb{Z}_{\geq 1}$ , the set  $\Sigma_k^{(a)} = \{aq^{k-2i-1}\}_{i=0,\dots,k-1}$  is called a *string*. Two strings  $\Sigma_k^{(a)}$  and  $\Sigma_{k'}^{(a')}$  are said to be in *general position* if the union  $\Sigma_k^{(a)} \cup \Sigma_{k'}^{(a')}$  is not a string or  $\Sigma_k^{(a)} \subset \Sigma_{k'}^{(a')}$  or  $\Sigma_{k'}^{(a')} \subset \Sigma_k^{(a)}$ .

Denote by  $L(m_+)$  the irreducible  $U_q \hat{\mathfrak{sl}}_2$ -module with highest weight monomial  $m_+$ . Let  $m_+ \neq 1$  and  $\in \mathbb{Z}[Y_a]_{a \in \mathbb{C}^\times}$  be a dominant monomial. Then  $m_+$  can be uniquely (up to permutation) written in the form

$$m_+ = \prod_{i=1}^s \left( \prod_{b \in \Sigma_{k_i}^{(a_i)}} Y_b \right),$$

where  $s$  is an integer,  $\Sigma_{k_i}^{(a_i)}, i = 1, \dots, s$ , are strings which are pairwise in general position and

$$L(m_+) = \bigotimes_{i=1}^s W_{k_i}^{(a_i)}, \quad \chi_q(m_+) = \prod_{i=1}^s \chi_q(W_{k_i}^{(a_i)}).$$

For  $j \in I$ , let

$$\beta_j : \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times} \rightarrow \mathbb{Z}[Y_a^{\pm 1}]_{a \in \mathbb{C}^\times}$$

be the ring homomorphism which sends, for all  $a \in \mathbb{C}^\times$ ,  $Y_{k,a} \mapsto 1$  for  $k \neq j$  and  $Y_{j,a} \mapsto Y_a$ .

Let  $V$  be a  $U_q \hat{\mathfrak{g}}$ -module. Then  $\beta_i(\chi_q(V))$ ,  $i = 1, 2$ , is the  $q$ -character of  $V$  considered as a  $U_{q_i}(\hat{\mathfrak{sl}}_2)$ -module.

In some situation, we can use the  $q$ -characters of  $U_q \hat{\mathfrak{sl}}_2$ -modules to compute the  $q$ -characters of  $U_q \hat{\mathfrak{g}}$ -modules for arbitrary  $\mathfrak{g}$ , see Section 5 in [FM01]. The corresponding algorithm is called the FM algorithm. The FM algorithm recursively computes the minimal possible  $q$ -character which contains  $m_+$  and is consistent when restricted to  $U_{q_i}(\hat{\mathfrak{sl}}_2)$ ,  $i = 1, 2$ . For example, if a module  $L(m_+)$  is special, then the FM algorithm applied to  $m_+$ , see [FM01], produces the correct  $q$ -character  $\chi_q(m_+)$ .

**2.6. Truncated  $q$ -characters.** We use the truncated  $q$ -characters ([HL10], [MY11b]). Given a set of monomials  $\mathcal{R} \subset \mathcal{P}$ , let  $\mathbb{Z}\mathcal{R} \subset \mathbb{Z}\mathcal{P}$  denote the  $\mathbb{Z}$ -module of formal linear combinations of elements of  $\mathcal{R}$  with integer coefficients. Define

$$\text{trunc}_{\mathcal{R}} : \mathcal{P} \rightarrow \mathcal{R}; \quad m \mapsto \begin{cases} m & \text{if } m \in \mathcal{R}, \\ 0 & \text{if } m \notin \mathcal{R}, \end{cases}$$

and extend  $\text{trunc}_{\mathcal{R}}$  as a  $\mathbb{Z}$ -module map  $\mathbb{Z}\mathcal{P} \rightarrow \mathbb{Z}\mathcal{R}$ .

Given a subset  $U \subset I \times \mathbb{C}^\times$ , let  $\mathcal{Q}_U$  be the subgroups of  $\mathcal{Q}$  generated by  $A_{i,a}$  with  $(i, a) \in U$ . Let  $\mathcal{Q}_U^\pm$  be the monoid generated by  $A_{i,a}^{\pm 1}$  with  $(i, a) \in U$ . We call  $\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  the  $q$ -character of  $L(m_+)$  truncated to  $U$ .

If  $U = I \times \mathbb{C}^\times$ , then  $\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  is the ordinary  $q$ -character of  $L(m_+)$ .

In some cases, the truncated  $q$ -characters can be computed using the following theorem.

**Theorem 2.2** ( Theorem 2.1, [MY11b] ). *Let  $U \subset I \times \mathbb{C}^\times$  and  $m_+ \in \mathcal{P}^+$ . Suppose that  $\mathcal{M} \subset \mathcal{P}$  is a finite set of distinct monomials such that*

- (i)  $\mathcal{M} \subset m_+ \mathcal{Q}_U^-$ ,
- (ii)  $\mathcal{P}^+ \cap \mathcal{M} = \{m_+\}$ ,
- (iii) for all  $m \in \mathcal{M}$  and all  $(i, a) \in U$ , if  $mA_{i,a}^{-1} \notin \mathcal{M}$ , then  $mA_{i,a}^{-1}A_{j,b} \notin \mathcal{M}$  unless  $(j, b) = (i, a)$ ,
- (iv) for all  $m \in \mathcal{M}$  and all  $i \in I$ , there exists a unique  $i$ -dominant monomial  $M \in \mathcal{M}$  such that

$$\text{trunc}_{\beta_i(M \mathcal{Q}_U^-)} \chi_q(\beta_i(M)) = \sum_{m' \in m \mathcal{Q}_{\{i\} \times \mathbb{C}^\times}} \beta_i(m').$$

Then

$$\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m.$$

Here by  $\chi_q(\beta_i(M))$  we mean the  $q$ -character of the irreducible  $U_{q_i}(\hat{\mathfrak{sl}}_2)$ -module with highest weight monomial  $\beta_i(M)$  and by  $\text{trunc}_{\beta_i}(M\mathcal{Q}_U^-)$  we mean keeping only the monomials of  $\chi_q(\beta_i(M))$  in the set  $\beta_i(M\mathcal{Q}_U^-)$ .

### 3. MAIN RESULTS

**3.1. First examples.** Without loss of generality, we fix  $a \in \mathbb{C}^\times$  and consider modules  $V$  with  $\mathcal{M}(V) \subset \mathbb{Z}[Y_{i,aq^k}]_{i \in I, k \in \mathbb{Z}}$ . In the following, for simplicity we write  $i_s, i_s^{-1}$  ( $s \in \mathbb{Z}$ ) instead of  $Y_{i,aq^s}, Y_{i,aq^s}^{-1}$  respectively. The  $q$ -characters of fundamental modules are easy to compute by using the FM algorithm.

**Lemma 3.1.** *The fundamental  $q$ -characters for  $U_q\hat{\mathfrak{g}}$  of type  $G_2$  are given by*

$$\begin{aligned}\chi_q(1_0) &= 1_0 + 1_2^{-1}2_1 + 1_41_62_7^{-1} + 1_41_8^{-1} + 1_6^{-1}1_8^{-1}2_5 + 1_{10}2_{11}^{-1} + 1_{12}^{-1}, \\ \chi_q(2_0) &= 2_0 + 1_11_31_52_6^{-1} + 1_11_31_7^{-1} + 1_11_5^{-1}1_7^{-1}2_4 + 1_3^{-1}1_5^{-1}1_7^{-1}2_22_4 \\ &\quad + 1_11_92_{10}^{-1} + 2_42_8^{-1} + 1_3^{-1}1_92_22_{10}^{-1} + 1_51_71_92_8^{-1}2_{10}^{-1} + 1_11_{11}^{-1} \\ &\quad + 1_3^{-1}1_{11}^{-1}2_2 + 1_51_71_{11}^{-1}2_8^{-1} + 1_51_9^{-1}1_{11}^{-1} + 1_7^{-1}1_9^{-1}1_{11}^{-1}2_6 + 2_{12}^{-1}. \quad \square\end{aligned}$$

For  $s \in \mathbb{Z}$ ,  $\chi_q(1_s)$  and  $\chi_q(2_s)$  are obtained by shift all indices by  $s$  in  $\chi_q(1_0)$  and  $\chi_q(2_0)$  respectively.

It is convenient to keep in mind the following lemma.

**Lemma 3.2.** *If  $b \in \mathbb{Z} \setminus \{\pm 2, \pm 8, \pm 12\}$ , then*

$$L(1_01_b) = L(1_0) \otimes L(1_b), \quad \dim L(1_01_b) = 49.$$

*If  $b \in \mathbb{Z} \setminus \{\pm 6, \pm 8, \pm 10, \pm 12\}$ , then*

$$L(2_02_b) = L(2_0) \otimes L(2_b), \quad \dim L(2_02_b) = 225.$$

*If  $b \in \mathbb{Z} \setminus \{\pm 7, \pm 11\}$ , then*

$$L(1_02_b) = L(1_0) \otimes L(2_b), \quad L(2_01_b) = L(2_0) \otimes L(1_b), \quad \dim L(1_02_b) = \dim L(2_01_b) = 105.$$

*In addition, we have*

$$\begin{aligned}\dim L(1_01_2) &= 34, \dim L(1_01_8) = 42, \dim L(1_01_{12}) = 48, \\ \dim L(2_02_6) &= 92, \dim L(2_02_8) = 210, \dim L(2_02_{10}) = 183, \dim L(2_02_{12}) = 224, \\ \dim L(1_02_7) &= \dim L(2_01_7) = 71, \dim L(1_02_{11}) = \dim L(2_01_{11}) = 98.\end{aligned}$$

*Proof.* By Lemma 3.1, the tensor products in the first three cases of the lemma are special. Therefore the tensor products are irreducible. Hence the first three cases of the lemma are true. The last part of the lemma can be proved using the methods of Section 5. In fact some of the dimensions follow from Theorem 8.1. We do not use this lemma in the proofs. Therefore we omit the details of the proof.  $\square$



**3.2. Definition of the modules  $\mathcal{B}_{k,\ell}^{(s)}, \mathcal{C}_{k,\ell}^{(s)}, \mathcal{D}_{k,\ell}^{(s)}, \mathcal{E}_{k,\ell}^{(s)}, \mathcal{F}_{k,\ell}^{(s)}$ .** For  $s \in \mathbb{Z}$ ,  $k, \ell \in \mathbb{Z}_{\geq 0}$ , define the following monomials.

$$\begin{aligned} B_{k,\ell}^{(s)} &= \left( \prod_{i=0}^{k-1} 2_{s+6i} \right) \left( \prod_{i=0}^{\ell-1} 1_{s+6k+2i+1} \right), \quad C_{k,\ell}^{(s)} = \left( \prod_{i=0}^{k-1} 2_{s+6i} \right) \left( \prod_{i=0}^{\ell-1} 2_{s+6k+6i+4} \right), \\ D_{k,\ell}^{(s)} &= \left( \prod_{i=0}^{k-1} 2_{s+6i} \right) 1_{s+6k+1} \left( \prod_{i=0}^{\ell-1} 2_{s+6k+6i+8} \right), \quad F_{k,\ell}^{(s)} = \left( \prod_{i=0}^{k-1} 1_{s+2i} \right) \left( \prod_{i=0}^{\ell-1} 1_{s+2k+2i+6} \right), \\ E_{k,\ell}^{(s)} &= \left( \prod_{i=0}^{k-1} 1_{s+2i} \right) \left( \prod_{i=0}^{\lfloor \frac{\ell-1}{2} \rfloor} 2_{s+2k+6i+3} \right) \left( \prod_{i=0}^{\lfloor \frac{\ell-2}{2} \rfloor} 2_{s+2k+6i+5} \right). \end{aligned}$$

Note that, in particular, for  $k \in \mathbb{Z}_{\geq 0}$ ,  $s \in \mathbb{Z}$ , we have the following trivial relations

$$\mathcal{B}_{k,0}^{(s)} = \mathcal{C}_{k,0}^{(s)} = \mathcal{C}_{0,k}^{(s-4)}, \quad \mathcal{D}_{k,0}^{(s)} = \mathcal{B}_{k,1}^{(s)}, \quad \mathcal{E}_{k,0}^{(s)} = \mathcal{B}_{0,k}^{(s-1)} = \mathcal{F}_{0,k}^{(s-6)} = \mathcal{F}_{k,0}^{(s)}. \quad (3.1)$$

Denote by  $\mathcal{B}_{k,\ell}^{(s)}, \mathcal{C}_{k,\ell}^{(s)}, \mathcal{D}_{k,\ell}^{(s)}, \mathcal{E}_{k,\ell}^{(s)}, \mathcal{F}_{k,\ell}^{(s)}$  the irreducible finite-dimensional highest  $l$ -weight  $U_q \hat{\mathfrak{g}}$ -modules with highest  $l$ -weight  $B_{k,\ell}^{(s)}, C_{k,\ell}^{(s)}, D_{k,\ell}^{(s)}, E_{k,\ell}^{(s)}, F_{k,\ell}^{(s)}$  respectively.

Note that  $\mathcal{B}_{k,\ell}^{(s)}, \mathcal{D}_{0,\ell}^{(s)}, \mathcal{D}_{k,0}^{(s)}$  are minimal affinizations. The modules  $\mathcal{B}_{0,\ell}^{(s)}, \mathcal{C}_{0,\ell}^{(s)}, \mathcal{F}_{0,\ell}^{(s)}, \mathcal{B}_{k,0}^{(s)}, \mathcal{C}_{k,0}^{(s)}, \mathcal{E}_{k,0}^{(s)}, \mathcal{F}_{k,0}^{(s)}$  are Kirillov-Reshetikhin modules.

Our first result is

**Theorem 3.3.** *The modules  $\mathcal{B}_{k,\ell}^{(s)}, \mathcal{C}_{k,\ell}^{(s)}, \mathcal{D}_{k,\ell}^{(s)}, \mathcal{E}_{k,\ell}^{(s)}, \mathcal{F}_{k,\ell}^{(s)}$ ,  $s \in \mathbb{Z}$ ,  $k, \ell \in \mathbb{Z}_{\geq 0}$ , are special. In particular, the FM algorithm works for all these modules.*

We prove Theorem 3.3 in Section 4. Note that the case of  $\mathcal{B}_{k,\ell}^{(s)}$  has been proved in Theorem 3.8 of [Her07]. In general, the modules in Theorem 3.3 are not thin. For example,  $\chi_q(1_0 1_2)$  has monomial  $1_4 1_6 1_8^{-1} 1_{10}^{-1}$  with coefficient 2.

**3.3. Extended T-system.** It is known that Kirillov-Reshetikhin modules  $\mathcal{B}_{k,0}^{(s)}, \mathcal{B}_{0,\ell}^{(s)}$  satisfy the following T-system relations, see [KR90],

$$[\mathcal{B}_{0,\ell}^{(s)}][\mathcal{B}_{0,\ell}^{(s+2)}] = [\mathcal{B}_{0,\ell+1}^{(s)}][\mathcal{B}_{0,\ell-1}^{(s+2)}] + [\mathcal{B}_{\lfloor \frac{\ell+2}{3} \rfloor, 0}^{(s+1)}][\mathcal{B}_{\lfloor \frac{\ell+1}{3} \rfloor, 0}^{(s+3)}][\mathcal{B}_{\lfloor \frac{\ell}{3} \rfloor, 0}^{(s+5)}], \quad (3.2)$$

$$[\mathcal{B}_{k,0}^{(s)}][\mathcal{B}_{k,0}^{(s+6)}] = [\mathcal{B}_{k+1,0}^{(s)}][\mathcal{B}_{k-1,0}^{(s+6)}] + [\mathcal{B}_{0,3k}^{(s+1)}], \quad (3.3)$$

where  $s \in \mathbb{Z}$ ,  $k, \ell \in \mathbb{Z}_{\geq 1}$ .

Our main result is

**Theorem 3.4.** *For  $s \in \mathbb{Z}$  and  $k, \ell \in \mathbb{Z}_{\geq 1}$ ,  $t \in \mathbb{Z}_{\geq 2}$ , we have the following relations in  $\text{Rep}(U_q \hat{\mathfrak{g}})$ .*

$$[\mathcal{B}_{k,\ell-1}^{(s)}][\mathcal{B}_{k-1,\ell}^{(s+6)}] = [\mathcal{B}_{k,\ell}^{(s)}][\mathcal{B}_{k-1,\ell-1}^{(s+6)}] + [\mathcal{E}_{3k-1, \lfloor \frac{2\ell-2}{3} \rfloor}^{(s+1)}][\mathcal{B}_{\lfloor \frac{\ell-1}{3} \rfloor, 0}^{(s+6k+6)}], \quad (3.4)$$

$$[\mathcal{E}_{0,\ell}^{(s)}] = [\mathcal{B}_{\lfloor \frac{\ell+1}{2} \rfloor, 0}^{(s+3)}][\mathcal{B}_{\lfloor \frac{\ell}{2} \rfloor, 0}^{(s+5)}], \quad (3.5)$$

$$[\mathcal{E}_{1,\ell}^{(s)}] = [\mathcal{D}_{0, \lfloor \frac{\ell}{2} \rfloor}^{(s-1)}][\mathcal{B}_{\lfloor \frac{\ell+1}{2} \rfloor, 0}^{(s+5)}], \quad (3.6)$$

$$[\mathcal{E}_{t,\ell-1}^{(s)}][\mathcal{E}_{t-1,\ell}^{(s+2)}] = [\mathcal{E}_{t,\ell}^{(s)}][\mathcal{E}_{t-1,\ell-1}^{(s+2)}] + \begin{cases} [\mathcal{D}_{r,p-1}^{(s+1)}][\mathcal{B}_{r+p,0}^{(s+3)}][\mathcal{B}_{r,3p-2}^{(s+5)}] & \text{if } t = 3r + 2, \ell = 2p - 1, \\ [\mathcal{B}_{r+p+1,0}^{(s+1)}][\mathcal{C}_{r,p}^{(s+3)}][\mathcal{B}_{r,3p-1}^{(s+5)}] & \text{if } t = 3r + 2, \ell = 2p, \\ [\mathcal{B}_{r+1,3p-2}^{(s+1)}][\mathcal{D}_{r,p-1}^{(s+3)}][\mathcal{B}_{r+p,0}^{(s+5)}] & \text{if } t = 3r + 3, \ell = 2p - 1, \\ [\mathcal{B}_{r+1,3p-1}^{(s+1)}][\mathcal{B}_{r+p+1,0}^{(s+3)}][\mathcal{C}_{r,p}^{(s+5)}] & \text{if } t = 3r + 3, \ell = 2p, \\ [\mathcal{B}_{r+p+1,0}^{(s+1)}][\mathcal{B}_{r+1,3p-2}^{(s+3)}][\mathcal{D}_{r,p-1}^{(s+5)}] & \text{if } t = 3r + 4, \ell = 2p - 1, \\ [\mathcal{C}_{r+1,p}^{(s+1)}][\mathcal{B}_{r+1,3p-1}^{(s+3)}][\mathcal{B}_{r+p+1,0}^{(s+5)}] & \text{if } t = 3r + 4, \ell = 2p, \end{cases} \quad (3.7)$$

$$[\mathcal{C}_{k,\ell-1}^{(s)}][\mathcal{C}_{k-1,\ell}^{(s+6)}] = [\mathcal{C}_{k,\ell}^{(s)}][\mathcal{C}_{k-1,\ell-1}^{(s+6)}] + [\mathcal{F}_{3k-2,3\ell-2}^{(s+1)}], \quad (3.8)$$

$$[\mathcal{D}_{0,\ell-1}^{(s)}][\mathcal{B}_{\ell,0}^{(s+8)}] = [\mathcal{D}_{0,\ell}^{(s)}][\mathcal{B}_{\ell-1,0}^{(s+8)}] + [\mathcal{B}_{0,3\ell-1}^{(s+4)}], \quad (3.9)$$

$$[\mathcal{D}_{k,\ell-1}^{(s)}][\mathcal{D}_{k-1,\ell}^{(s+6)}] = [\mathcal{D}_{k,\ell}^{(s)}][\mathcal{D}_{k-1,\ell-1}^{(s+6)}] + [\mathcal{F}_{3k-1,3\ell-1}^{(s+1)}], \quad (3.10)$$

$$[\mathcal{F}_{k,\ell-1}^{(s)}][\mathcal{F}_{k-1,\ell}^{(s+2)}] = [\mathcal{F}_{k,\ell}^{(s)}][\mathcal{F}_{k-1,\ell-1}^{(s+2)}] + \begin{cases} [\mathcal{B}_{r,0}^{(s+1)}][\mathcal{D}_{r,\lfloor \frac{\ell}{3} \rfloor}^{(s+3)}][\mathcal{C}_{r,\lfloor \frac{\ell+1}{3} \rfloor}^{(s+5)}][\mathcal{B}_{\lfloor \frac{\ell-1}{3} \rfloor,0}^{(s+2k+11)}] & \text{if } k = 3r + 1, \\ [\mathcal{C}_{r+1,\lfloor \frac{\ell+1}{3} \rfloor}^{(s+1)}][\mathcal{B}_{r,0}^{(s+3)}][\mathcal{D}_{r,\lfloor \frac{\ell}{3} \rfloor}^{(s+5)}][\mathcal{B}_{\lfloor \frac{\ell-1}{3} \rfloor,0}^{(s+2k+11)}] & \text{if } k = 3r + 2, \\ [\mathcal{D}_{r+1,\lfloor \frac{\ell}{3} \rfloor}^{(s+1)}][\mathcal{C}_{r+1,\lfloor \frac{\ell+1}{3} \rfloor}^{(s+3)}][\mathcal{B}_{r,0}^{(s+5)}][\mathcal{B}_{\lfloor \frac{\ell-1}{3} \rfloor,0}^{(s+2k+11)}] & \text{if } k = 3r + 3. \end{cases} \quad (3.11)$$

We prove Theorem 3.4 in Section 5.

Note that since  $\mathcal{D}_{k,0}^{(s)} = \mathcal{B}_{k,1}^{(s)}$ , equations for  $\mathcal{D}_{k,0}^{(s)}$  are included in the equations for  $\mathcal{B}_{k,1}^{(s)}$ .

All relations except (3.5), (3.6) in Theorem 3.4 are written in the form  $[\mathcal{L}][\mathcal{R}] = [\mathcal{T}][\mathcal{B}] + [\mathcal{S}]$ , where  $\mathcal{L}, \mathcal{R}, \mathcal{T}, \mathcal{B}$  are irreducible modules which we call *left, right, top and bottom modules* and  $\mathcal{S}$  is a tensor product of some irreducible modules. We call the factors of  $\mathcal{S}$  *sources*. Moreover, we have the following theorem.

**Theorem 3.5.** *For each relation in Theorem 3.4, all summands on the right hand side,  $\mathcal{T} \otimes \mathcal{B}$  and  $\mathcal{S}$ , are irreducible.*

We will prove Theorem 3.5 in Section 6.

Recall that the  $q$ -characters of modules for different  $s$  are related by the simple shift of indexes, see (2.6).

We have the following proposition.

**Proposition 3.6.** *Given  $\chi_q(1_s), \chi_q(2_s)$ , one can obtain the  $q$ -characters of  $\mathcal{B}_{k,\ell}^{(s)}, \mathcal{C}_{k,\ell}^{(s)}, \mathcal{D}_{k,\ell}^{(s)}, \mathcal{E}_{k,\ell}^{(s)}, \mathcal{F}_{k,\ell}^{(s)}$ ,  $s \in \mathbb{Z}$ ,  $k, \ell \in \mathbb{Z}_{\geq 0}$ , recursively, by using (3.1), and computing the  $q$ -character of the top module through the  $q$ -characters of other modules in relations in Theorem 3.4.*

*Proof.* **Claim 1.** Let  $n, m$  be positive integers. Then the  $q$ -characters

$$\begin{aligned} \chi_q(\mathcal{B}_{k,\ell}^{(s)}), \quad k \leq n, \ell \leq m, \quad \chi_q(\mathcal{C}_{k,\ell}^{(s)}), \quad k \leq n-1, \ell \leq \lceil \frac{2m+1}{6} \rceil, \\ \chi_q(\mathcal{D}_{k,\ell}^{(s)}), \quad k \leq n-1, \ell \leq \lceil \frac{2m+1}{6} \rceil, \quad \chi_q(\mathcal{E}_{k,\ell}^{(s)}), \quad k \leq 3n-1, \ell \leq \lceil \frac{2m-2}{3} \rceil, \\ \chi_q(\mathcal{F}_{k,\ell}^{(s)}), \quad k \leq 3n-4, \ell \leq m+2, \end{aligned}$$

can be computed recursively starting from  $\chi_q(1_0), \chi_q(2_0)$ .

We use induction on  $n, m$  to prove Claim 1. For simplicity, we do not write the upper-subscripts "(s)" in the remaining part of the proof. We know that, see [Her06], the  $q$ -characters of Kirillov-Reshetikhin modules can be computed from  $\chi_q(1_0), \chi_q(2_0)$ .

When  $n = 0, m = 1$ , Claim 1 is clearly true. It is clear that  $\chi_q(\mathcal{D}_{0,1})$  can be computed using (3.9). Therefore Claim 1 holds for  $n = 1, m = 0$ ,

Suppose that for  $n \leq n_1$  and  $m \leq m_1$ , Claim 1 is true. Let  $n = n_1 + 1, m = m_1$ . We need to show that Claim 1 is true. Then we need to show that

$$\begin{aligned} \chi_q(\mathcal{B}_{n_1+1,\ell}), \quad \ell \leq m_1, \quad \chi_q(\mathcal{C}_{n_1,\ell}), \quad \ell \leq \lceil \frac{2m_1+1}{6} \rceil, \quad \chi_q(\mathcal{D}_{n_1,\ell}), \quad \ell \leq \lceil \frac{2m_1+1}{6} \rceil, \\ \chi_q(\mathcal{E}_{k,\ell}), \quad k = 3n_1, 3n_1+1, 3n_1+2, \quad \ell \leq \lceil \frac{2m_1-2}{3} \rceil, \\ \chi_q(\mathcal{F}_{k,\ell}), \quad k = 3n_1-3, 3n_1-2, 3n_1-1, \quad \ell \leq m_1+2, \end{aligned}$$

can be computed.

We compute the following modules

$$\begin{aligned} \chi_q(\mathcal{F}_{3n_1-3,\ell}), \quad \ell \leq m_1+2, \quad \chi_q(\mathcal{F}_{3n_1-2,\ell}), \quad \ell \leq m_1+2, \quad \chi_q(\mathcal{C}_{n_1,\ell}), \quad \ell \leq \lfloor \frac{m_1+3}{3} \rfloor, \\ \chi_q(\mathcal{F}_{3n_1-1,\ell}), \quad \ell \leq m_1+2, \quad \chi_q(\mathcal{D}_{n_1,\ell}), \quad \ell \leq \lceil \frac{2m_1+1}{6} \rceil, \quad \chi_q(\mathcal{C}_{n_1,\ell}), \quad \ell \leq \lceil \frac{2m_1+1}{6} \rceil, \\ \chi_q(\mathcal{E}_{3n_1,\ell}), \quad \ell \leq \lceil \frac{2m_1-2}{3} \rceil, \quad \chi_q(\mathcal{E}_{3n_1,\ell}), \quad \ell \leq \lceil \frac{2m_1-2}{3} \rceil, \quad \chi_q(\mathcal{E}_{3n_1+1,\ell}), \quad \ell \leq \lceil \frac{2m_1-2}{3} \rceil, \\ \chi_q(\mathcal{E}_{3n_1+2,\ell}), \quad \ell \leq \lceil \frac{2m_1-2}{3} \rceil, \quad \chi_q(\mathcal{B}_{n_1+1,\ell}), \quad \ell \leq m_1, \end{aligned}$$

in the order as shown. At each step, we consider the module that we want to compute as a top module and use the corresponding relation in Theorem 3.4 and known  $q$ -characters. For example, we consider the first set of modules  $\chi_q(\mathcal{F}_{3n_1-3,\ell}), \ell \leq m_1+2$ . Since  $\lfloor \frac{m_1+3}{3} \rfloor \leq \lceil \frac{2m_1+1}{6} \rceil$ ,  $\chi_q(\mathcal{C}_{n_1-1,\ell}), \ell \leq \lfloor \frac{m_1+3}{3} \rfloor$ , is known by induction hypothesis. Similarly,  $\chi_q(\mathcal{D}_{n_1-1,\ell}), \ell \leq \lfloor \frac{m_1+2}{3} \rfloor$  is known. Therefore  $\chi_q(\mathcal{F}_{3n_1-3,\ell}), \ell \leq m_1+2$ , is computed using the last equation of (3.11).

Similarly, we show that Claim 1 holds for  $n = n_1, m = m_1 + 1$ . Therefore Claim 1 is true for all  $n \geq 1, m \geq 1$ .  $\square$

#### 4. PROOF OF THEOREM 3.3

In this section, we will show that the modules  $\mathcal{B}_{k,\ell}^{(s)}, \mathcal{C}_{k,\ell}^{(s)}, \mathcal{D}_{k,\ell}^{(s)}, \mathcal{E}_{k,\ell}^{(s)}, \mathcal{F}_{k,\ell}^{(s)}$  are special.

Since  $\mathcal{B}_{0,\ell}^{(s)}, \mathcal{C}_{0,\ell}^{(s)}, \mathcal{F}_{0,\ell}^{(s)}, \mathcal{B}_{k,0}^{(s)}, \mathcal{C}_{k,0}^{(s)}, \mathcal{E}_{k,0}^{(s)}, \mathcal{F}_{k,0}^{(s)}$  are Kirillov-Reshetikhin modules, they are special.

**4.1. The case of  $C_{k,\ell}^{(s)}$ .** Let  $m_+ = C_{k,\ell}^{(s)}$  with  $k, \ell \in \mathbb{Z}_{\geq 1}$ . Without loss of generality, we can assume that  $s = 6$ . Then

$$m_+ = (2_6 2_{12} \cdots 2_{6k})(2_{6k+10} 2_{6k+16} \cdots 2_{6k+6\ell+4}).$$

**Case 1.**  $k = 1$ . Let  $U = I \times \{aq^s : s \in \mathbb{Z}, s < 6\ell + 13\}$ . Clearly, all monomials in  $\chi_q(m_+) - \text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  are right-negative. Therefore it is sufficient to show that  $\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  is special.

Let  $\mathcal{M}$  be the finite set consisting of the following monomials

$$m_0 = m_+, \quad m_1 = m_0 A_{2,9}^{-1}, \quad m_2 = m_1 A_{1,12}^{-1}, \quad m_3 = m_2 A_{1,10}^{-1}, \quad m_4 = m_3 A_{1,8}^{-1}, \quad m_5 = m_4 A_{2,11}^{-1}.$$

It is clear that  $\mathcal{M}$  satisfies the conditions in Theorem 2.2. Therefore

$$\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and  $\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  is special.

**Case 2.**  $k > 1$ . Since the conditions of Theorem 2.2 do not apply to this case, we use another technique to show that  $L(m_+)$  is special. We embed  $L(m_+)$  into two different tensor products. In both tensor products, each factor is special. Therefore we can use the FM algorithm to compute the  $q$ -characters of the factors. We classify the dominant monomials in the first tensor product and show that the only dominant monomial in the first tensor product which occurs in the second tensor product is  $m_+$  which proves that  $L(m_+)$  is special.

The first tensor product is  $L(m'_1) \otimes L(m'_2)$ , where

$$m'_1 = 2_6 2_{12} \cdots 2_{6k}, \quad m'_2 = 2_{6k+10} 2_{6k+16} \cdots 2_{6k+6\ell+4}.$$

We use the FM algorithm to compute  $\chi_q(m'_1), \chi_q(m'_2)$  and classify all dominant monomials in  $\chi_q(m'_1) \chi_q(m'_2)$ . Let  $m = m_1 m_2$  be a dominant monomial, where  $m_i \in \chi_q(m'_i)$ ,  $i = 1, 2$ . If  $m_2 \neq m'_2$ , then  $m$  is a right negative monomial therefore  $m$  is not dominant. Hence  $m_2 = m'_2$ .

If  $m_1 \neq m'_1$ , then  $m_1$  is right negative. Since  $m$  is dominant, each factor with a negative power in  $m_1$  needs to be canceled by a factor in  $m'_2$ . All possible cancellations cancel  $2_{6k+10}$  in  $m'_2$ . We have  $\mathcal{M}(L(m'_1)) \subset \mathcal{M}(\chi_q(2_6 2_{12} \cdots 2_{6k-6}) \chi_q(2_{6k}))$ . Only monomials in  $\chi_q(2_{6k})$  can cancel  $2_{6k+10}$ . These monomials are  $1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1}$ ,  $1_{6k+3}^{-1} 1_{6k+9} 2_{6k+2} 2_{6k+10}^{-1}$ , and  $1_{6k+5} 1_{6k+7} 1_{6k+9} 2_{6k+8}^{-1} 2_{6k+10}^{-1}$ . Therefore  $m_1$  is in one of the following polynomials

$$\chi_q(2_6 2_{12} \cdots 2_{6k-6}) 1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1}, \tag{4.1}$$

$$\chi_q(2_6 2_{12} \cdots 2_{6k-6}) 1_{6k+3}^{-1} 1_{6k+9} 2_{6k+2} 2_{6k+10}^{-1}, \tag{4.2}$$

$$\chi_q(2_6 2_{12} \cdots 2_{6k-6}) 1_{6k+5} 1_{6k+7} 1_{6k+9} 2_{6k+8}^{-1} 2_{6k+10}^{-1}. \tag{4.3}$$

**Subcase 2.1.** Let  $m_1$  be in (4.1). If  $m_1 = 2_6 2_{12} \cdots 2_{6k-6} 1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1}$ , then

$$m = m_1 m_2 = 2_6 2_{12} \cdots 2_{6k-6} 1_{6k+1} 1_{6k+9} 2_{6k+16} \cdots 2_{6k+6\ell+4} \tag{4.4}$$

is dominant. Suppose that

$$m_1 \neq 2_6 2_{12} \cdots 2_{6k-6} 1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1}.$$

Then  $m_1 = n_1 1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1}$ , where  $n_1$  is a non-highest monomial in  $\chi_q(2_6 2_{12} \cdots 2_{6k-6})$ . Since  $n_1$  is right negative,  $1_{6k+1}$  or  $1_{6k+9}$  should cancel a factor of  $n_1$  with a negative power. Using the FM algorithm, we see that there exists a factor  $1_{6k-1}^2$  or  $1_{6k+7}^2$  in a monomial in

$\chi_q(2_6 2_{12} \cdots 2_{6k-6}) 1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1}$ . By Lemma 3.1, neither  $1_{6k-1}^2$  nor  $1_{6k+7}^2$  appear. This is a contradiction.

**Subcase 2.2.** Let  $m_1$  be in (4.2). If  $m_1 = 2_6 2_{12} \cdots 2_{6k-6} 1_{6k+3}^{-1} 1_{6k+9} 2_{6k+2} 2_{6k+10}^{-1}$ , then  $m = m_1 m_2$  is not dominant. Suppose that  $m_1 \neq 2_6 2_{12} \cdots 2_{6k-6} 1_{6k+3}^{-1} 1_{6k+9} 2_{6k+2} 2_{6k+10}^{-1}$ . Then  $m_1 = n_1 1_{6k+3}^{-1} 1_{6k+9} 2_{6k+2} 2_{6k+10}^{-1}$ , where  $n_1$  is a non-highest monomial in  $\chi_q(2_6 2_{12} \cdots 2_{6k-6})$ . Since  $n_1$  is right negative,  $1_{6k+9}$  or  $2_{6k+2}$  should cancel a factor of  $n_1$  with a negative power. Using the FM algorithm, we see that there exists either a factor  $1_{6k+7}^2$  or a factor  $2_{6k-4}^2$  in a monomial in  $\chi_q(2_6 2_{12} \cdots 2_{6k-6}) 1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1}$ . By Lemma 3.1, neither  $1_{6k+7}^2$  nor  $2_{6k-4}^2$  appear. This is a contradiction.

**Subcase 2.3.** Let  $m_1$  be in (4.3). If  $m_1 = 2_6 2_{12} \cdots 2_{6k-6} 1_{6k+5} 1_{6k+7} 1_{6k+9} 2_{6k+8}^{-1} 2_{6k+10}^{-1}$ , then  $m = m_1 m_2$  is not dominant. Suppose that  $m_1 \neq 2_6 2_{12} \cdots 2_{6k-6} 1_{6k+5} 1_{6k+7} 1_{6k+9} 2_{6k+8}^{-1} 2_{6k+10}^{-1}$ . Then we have  $m_1 = n_1 1_{6k+5} 1_{6k+7} 1_{6k+9} 2_{6k+8}^{-1} 2_{6k+10}^{-1}$ , where  $n_1$  is a non-highest monomial in  $\chi_q(2_6 2_{12} \cdots 2_{6k-6})$ . Since  $n_1$  is right negative,  $1_{6k+5}$  or  $1_{6k+7}$  or  $1_{6k+9}$  should cancel a factor of  $n_1$  with a negative power. Using the FM algorithm, we see that there exists a factor  $1_{6k+7}$  or  $1_{6k+5}$  or  $1_{6k+3}^2$  in a monomial in  $\chi_q(2_6 2_{12} \cdots 2_{6k-6}) 1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1}$ . By Lemma 3.1,  $1_{6k+7}$ ,  $1_{6k+5}$ , and  $1_{6k+3}^2$  do not appear. This is a contradiction.

Therefore the only dominant monomials in  $\chi_q(m'_1) \chi_q(m'_2)$  are  $m_+$  and (4.4).

The second tensor product is  $L(m''_1) \otimes L(m''_2)$ , where

$$m''_1 = 2_6 2_{12} \cdots 2_{6k-6}, \quad m''_2 = 2_{6k} 2_{6k+10} 2_{6k+16} \cdots 2_{6k+6\ell+4}.$$

The monomial (4.4) is

$$n = m_+ A_{2,6k+3}^{-1} A_{1,6k+6}^{-1} A_{1,6k+4}^{-1} A_{2,6k+7}^{-1}. \quad (4.5)$$

Since  $A_{i,a}, i \in I, a \in \mathbb{C}^\times$  are algebraically independent, the expression (4.5) of  $n$  of the form  $m_+ \prod_{i \in I, a \in \mathbb{C}^\times} A_{i,a}^{-v_{i,a}}$ , where  $v_{i,a}$  are some integers, is unique. Suppose that the monomial  $n$  is in  $\chi_q(m''_1) \chi_q(m''_2)$ . Then  $n = n_1 n_2$ , where  $n_i \in \chi_q(m''_i), i = 1, 2$ . By the expression (4.5), we have  $n_1 = m''_1$  and

$$n_2 = m''_2 A_{2,6k+3}^{-1} A_{1,6k+6}^{-1} A_{1,6k+4}^{-1} A_{2,6k+7}^{-1}.$$

By the FM algorithm, the monomial  $m''_2 A_{2,6k+3}^{-1} A_{1,6k+6}^{-1} A_{1,6k+4}^{-1} A_{2,6k+7}^{-1}$  is not in  $\chi_q(m''_2)$ . This contradicts the fact that  $n_2 \in \chi_q(m''_2)$ . Therefore  $n$  is not in  $\chi_q(m''_1) \chi_q(m''_2)$ .

**4.2. The case of  $\mathcal{B}_{k,\ell}^{(s)}$ .** Let  $m_+ = B_{k,\ell}^{(s)}$  with  $k, \ell \in \mathbb{Z}_{\geq 1}$ . Without loss of generality, we can assume that  $s = 6$ . Then

$$m_+ = (2_6 2_{12} \cdots 2_{6k}) (1_{6k+7} 1_{6k+9} \cdots 1_{6k+2\ell+5}).$$

Let  $U = I \times \{aq^s : s \in \mathbb{Z}, s < 6k + 2\ell + 6\}$ . Clearly, all monomials in the polynomial  $\chi_q(m_+) - \text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  are right-negative. Therefore it is sufficient to show that the truncated  $q$ -character  $\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  is special.

Let  $\mathcal{M}$  be the finite set consisting of the following monomials

$$m_0 = m_+, \quad m_1 = m_0 A_{2,6k+3}^{-1}, \quad m_2 = m_1 A_{2,6k-3}^{-1}, \quad \dots, \quad m_k = m_{k-1} A_{2,9}^{-1}.$$

It is clear that  $\mathcal{M}$  satisfies the conditions in Theorem 2.2. Therefore

$$\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and  $\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  is special.

**4.3. The case of  $\mathcal{D}_{k,\ell}^{(s)}$ .** Let  $m_+ = D_{k,\ell}^{(s)}$  with  $k, \ell \in \mathbb{Z}_{\geq 0}$ . Without loss of generality, we can assume that  $s = 0$ . Then

$$m_+ = (2_0 2_6 \cdots 2_{6k-6}) 1_{6k+1} (2_{6k+8} 2_{6k+14} \cdots 2_{6k+6\ell+2}).$$

**Case 1.**  $k = 0$ . Let  $U = I \times \{aq^s : s \in \mathbb{Z}, s < 6\ell + 5\}$ . Clearly, all monomials in  $\chi_q(m_+) - \text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  are right-negative. Therefore it is sufficient to show that  $\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  is special.

Let

$$M = \{m_+, m_+ A_{1,2}^{-1}\}.$$

It is clear that  $\mathcal{M}$  satisfies the conditions in Theorem 2.2. Therefore

$$\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and  $\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  is special.

**Case 2.**  $k > 0$ . Let

$$\begin{aligned} m'_1 &= 2_0 2_6 \cdots 2_{6k-6} 1_{6k+1}, \quad m'_2 = 2_{6k+8} 2_{6k+14} \cdots 2_{6k+6\ell+2}, \\ m''_1 &= 2_0 2_6 \cdots 2_{6k-6}, \quad m''_2 = 1_{6k+1} 2_{6k+8} 2_{6k+14} \cdots 2_{6k+6\ell+2}. \end{aligned}$$

Then  $\mathcal{M}(L(m_+)) \subset \mathcal{M}(\chi_q(m'_1) \chi_q(m'_2)) \cap \mathcal{M}(\chi_q(m''_1) \chi_q(m''_2))$ .

By using similar arguments as the case of  $\mathcal{C}_{k,\ell}^{(s)}$ , we show that the only dominant monomials in  $\chi_q(m'_1) \chi_q(m'_2)$  are  $m_+$  and

$$n = 2_0 2_6 \cdots 2_{6k-6} 1_{6k+5} 1_{6k+7} 2_{6k+14} 2_{6k+20} \cdots 2_{6k+6\ell+2} = m_+ A_{1,6k+2}^{-1} A_{2,6k+5}^{-1}.$$

Moreover,  $n$  is not in  $\chi_q(m''_1) \chi_q(m''_2)$ . Therefore the only dominant monomial in  $\chi_q(m_+)$  is  $m_+$ .

**4.4. The case of  $\mathcal{E}_{k,\ell}^{(s)}$ .** Let  $m_+ = E_{k,\ell}^{(s)}$  with  $k, \ell \in \mathbb{Z}_{\geq 0}$ . Without loss of generality, we can assume that  $s = 1$ . Suppose that  $\ell = 2r + 1, r \geq 0$  and  $k = 3p, p \geq 1$ . The cases of  $\ell = 2r, r \geq 1$ , or  $k = 0$  or  $k = 3p + 1, p \geq 0$  or  $k = 3p + 2, p \geq 0$  are similar.

Then

$$m = (1_1 1_3 \cdots 1_{6p-1}) (2_{6p+4} 2_{6p+10} \cdots 2_{6p+6r-2} 2_{6p+6r+4}) (2_{6p+6} 2_{6p+12} \cdots 2_{6p+6r}).$$

Let  $U = I \times \{aq^s : s \in \mathbb{Z}, s < 6p + 6r + 3\}$ . Clearly, all monomials in the polynomial  $\chi_q(m_+) - \text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  are right-negative. Therefore it is sufficient to show that the truncated  $q$ -character  $\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  is special.

Let  $\mathcal{M}$  be the finite set consisting of the following monomials

$$\begin{aligned} m_0 &= m_+, \quad m_1 = m_0 A_{1,6p}^{-1}, \quad m_2 = m_1 A_{1,6p-2}^{-1}, \quad \dots, \quad m_{3p} = m_{3p-1} A_{1,2}^{-1}, \\ m_{3p+1} &= m_{3p} A_{2,6p-4}^{-1}, \quad m_{3p+2} = m_{3p+1} A_{2,6p-10}^{-1}, \quad \dots, \quad m_{4p} = m_{4p-2} A_{2,6}^{-1}. \end{aligned}$$

It is clear that  $\mathcal{M}$  satisfies the conditions in Theorem 2.2. Therefore

$$\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and  $\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  is special.

**4.5. The case of  $\mathcal{F}_{k,\ell}^{(s)}$ .** Let  $m_+ = F_{k,\ell}^{(s)}$  with  $k, \ell \in \mathbb{Z}_{\geq 1}$ . Without loss of generality, we can assume that  $s = 1$ . Then

$$m_+ = (1_1 1_3 \cdots 1_{2k-1})(1_{2k+7} 1_{2k+9} \cdots 1_{2k+2\ell+5}).$$

**Case 1.**  $k = 1$ . Let  $U = I \times \{aq^s : s \in \mathbb{Z}, s < 2\ell + 8\}$ . Clearly, all monomials in  $\chi_q(m_+) - \text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  are right-negative. Therefore it is sufficient to show that  $\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  is special.

Let  $\mathcal{M}$  be the finite set consisting of the following monomials

$$m_0 = m_+, \quad m_1 = m_0 A_{1,2}^{-1}, \quad m_2 = m_1 A_{2,5}^{-1}.$$

It is clear that  $\mathcal{M}$  satisfies the conditions in Theorem 2.2. Therefore

$$\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and  $\text{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$  is special.

**Case 2.**  $k > 1$ . Let

$$\begin{aligned} m'_1 &= 1_1 1_3 \cdots 1_{2k-1}, \quad m'_2 = 1_{2k+7} 1_{2k+9} \cdots 1_{2k+2\ell+5}, \\ m''_1 &= 1_1 1_3 \cdots 1_{2k-3}, \quad m''_2 = 1_{2k-1} 1_{2k+7} 1_{2k+9} \cdots 1_{2k+2\ell+5}. \end{aligned}$$

Then  $\mathcal{M}(L(m_+)) \subset \mathcal{M}(\chi_q(m'_1)\chi_q(m'_2)) \cap \mathcal{M}(\chi_q(m''_1)\chi_q(m''_2))$ .

By using similar arguments as the case of  $\mathcal{C}_{k,\ell}^{(s)}$ , we can show that the only dominant monomials in  $\chi_q(m'_1)\chi_q(m'_2)$  are  $m_+$  and

$$\begin{aligned} n_1 &= 1_1 1_3 \cdots 1_{2k-3} 1_{2k+3} 1_{2k+9} 1_{2k+11} \cdots 1_{2k+2\ell+5} \\ &= m_+ A_{1,2k}^{-1} A_{2,2k+3}^{-1} A_{1,2k+6}^{-1}, \\ n_2 &= 1_1 1_3 \cdots 1_{2k-3} 1_{2k+7} 1_{2k+9} 1_{2k+13} 1_{2k+15} \cdots 1_{2k+2\ell+5} \\ &= n_1 A_{1,2k+4}^{-1} A_{2,2k+7}^{-1} A_{1,2k+10}^{-1}, \\ n_3 &= 1_1 1_3 \cdots 1_{2k-5} 1_{2k+7} 1_{2k+13} 1_{2k+15} \cdots 1_{2k+2\ell+5} \\ &= n_2 A_{1,2k-2}^{-1} A_{2,2k+1}^{-1} A_{1,2k+4}^{-1} A_{1,2k+2}^{-1} A_{2,2k+5}^{-1} A_{1,2k+8}^{-1}, \\ n_4 &= 1_1 1_3 \cdots 1_{2k-7} 1_{2k+13} 1_{2k+15} \cdots 1_{2k+2\ell+5} \\ &= n_3 A_{1,2k-4}^{-1} A_{2,2k-1}^{-1} A_{1,2k+2}^{-1} A_{1,2k}^{-1} A_{2,2k+3}^{-1} A_{1,2k+6}^{-1}. \end{aligned}$$

Moreover,  $n_1, n_2, n_3, n_4$  are not in  $\chi_q(m'_1)\chi_q(m'_2)$ . Therefore the only dominant monomial in  $\chi_q(m_+)$  is  $m_+$ .

## 5. PROOF OF THEOREM 3.4

We use the FM algorithm to classify dominant monomials in  $\chi_q(\mathcal{L})\chi_q(\mathcal{R})$ ,  $\chi_q(\mathcal{T})\chi_q(\mathcal{B})$ , and  $\chi_q(\mathcal{S})$ .

### 5.1. Classification of dominant monomials in $\chi_q(\mathcal{L})\chi_q(\mathcal{R})$ and $\chi_q(\mathcal{T})\chi_q(\mathcal{B})$ .

**Lemma 5.1.** *We have the following cases.*

(1) Let  $M = B_{k,\ell-1}^{(s)} B_{k-1,\ell}^{(s+6)}$ ,  $k \geq 1, \ell \geq 1$ . Then dominant monomials in  $\chi_q(\mathcal{B}_{k,\ell-1}^{(s)})\chi_q(\mathcal{B}_{k-1,\ell}^{(s+6)})$  are

$$\begin{aligned} M_0 &= M, \quad M_1 = M A_{1,s+6k+2\ell-2}^{-1}, \quad M_2 = M_1 A_{1,s+6k+2\ell-4}^{-1}, \quad \dots, \\ M_{\ell-1} &= M_{\ell-2} A_{1,s+6k+2}^{-1}, \quad M_\ell = M_{\ell-1} A_{2,s+6k-3}^{-1} A_{1,s+6k}^{-1}, \\ M_{\ell+1} &= M_\ell A_{2,s+6k-9}^{-1}, \quad M_{\ell+2} = M_{\ell+1} A_{2,s+6k-15}^{-1}, \quad \dots, \quad M_{k+\ell-1} = M_{k+\ell-2} A_{2,s+3}^{-1}. \end{aligned}$$

The dominant monomials in  $\chi_q(\mathcal{B}_{k,\ell}^{(s)})\chi_q(\mathcal{B}_{k-1,\ell-1}^{(s+6)})$  are  $M_0, \dots, M_{k+\ell-2}$ .

(2) Let  $M = C_{k,\ell-1}^{(s)} C_{k-1,\ell}^{(s+6)}$ ,  $k \geq 1, \ell \geq 1$ . Then dominant monomials in  $\chi_q(\mathcal{C}_{k,\ell-1}^{(s)})\chi_q(\mathcal{C}_{k-1,\ell}^{(s+6)})$  are

$$\begin{aligned} M_0 &= M, \quad M_1 = M A_{2,s+6k+6\ell-5}^{-1}, \quad M_2 = M_1 A_{2,s+6k+6\ell-11}^{-1}, \quad \dots, \\ M_{\ell-1} &= M_{\ell-2} A_{2,s+6k+7}^{-1}, \quad M_\ell = M_{\ell-1} A_{2,s+6k-3}^{-1} A_{1,s+6k}^{-1} A_{1,s+6k-2}^{-1} A_{2,s+6k+1}^{-1}, \\ M_{\ell+1} &= M_\ell A_{2,s+6k-9}^{-1}, \quad M_{\ell+2} = M_{\ell+1} A_{2,s+6k-15}^{-1}, \quad \dots, \quad M_{k+\ell-1} = M_{k+\ell-2} A_{2,s+3}^{-1}. \end{aligned}$$

The dominant monomials in  $\chi_q(\mathcal{C}_{k,\ell}^{(s)})\chi_q(\mathcal{C}_{k-1,\ell-1}^{(s+6)})$  are  $M_0, \dots, M_{k+\ell-2}$ .

(3) Let  $M = D_{0,\ell-1}^{(s)} B_{\ell,0}^{(s+8)}$ ,  $\ell \geq 1$ . Then dominant monomials in  $\chi_q(\mathcal{D}_{0,\ell-1}^{(s)})\chi_q(\mathcal{B}_{\ell,0}^{(s+8)})$  are

$$\begin{aligned} M_0 &= M, \quad M_1 = M A_{2,s+6\ell-1}^{-1}, \quad M_2 = M_1 A_{2,s+6\ell-7}^{-1}, \quad \dots, \\ M_{\ell-1} &= M_{\ell-2} A_{2,s+11}^{-1}, \quad M_\ell = M_{\ell-1} A_{1,s+6k+2}^{-1} A_{2,s+6k+5}^{-1}. \end{aligned}$$

The dominant monomials in  $\chi_q(\mathcal{D}_{0,\ell}^{(s)})\chi_q(\mathcal{B}_{\ell-1,0}^{(s+8)})$  are  $M_0, \dots, M_{\ell-1}$ .

(4) Let  $M = D_{k,\ell-1}^{(s)} D_{k-1,\ell}^{(s+6)}$ ,  $k \geq 1, \ell \geq 1$ . Then dominant monomials in  $\chi_q(\mathcal{D}_{k,\ell-1}^{(s)})\chi_q(\mathcal{D}_{k-1,\ell}^{(s+6)})$  are

$$\begin{aligned} M_0 &= M, \quad M_1 = M A_{2,s+6k+6\ell-1}^{-1}, \quad M_2 = M_1 A_{2,s+6k+6\ell-7}^{-1}, \quad \dots, \\ M_{\ell-1} &= M_{\ell-2} A_{2,s+6k+11}^{-1}, \quad M_\ell = M_{\ell-1} A_{1,s+6k+2}^{-1} A_{2,s+6k+5}^{-1}, \\ M_{\ell+1} &= M_\ell A_{2,s+6k-3}^{-1} A_{1,s+6k}^{-1}, \quad M_{\ell+2} = M_{\ell+1} A_{2,s+6k-9}^{-1}, \quad \dots, \quad M_{k+\ell} = M_{k+\ell-1} A_{2,s+3}^{-1}. \end{aligned}$$

The dominant monomials in  $\chi_q(\mathcal{D}_{k,\ell}^{(s)})\chi_q(\mathcal{D}_{k-1,\ell-1}^{(s+6)})$  are  $M_0, \dots, M_{k+\ell-1}$ .

(5) Let  $M = E_{k,\ell-1}^{(s)} E_{k-1,\ell}^{(s+2)}$ ,  $k \geq 1, \ell \geq 1$ .

If  $\ell = 2r + 1$ , then dominant monomials in  $\chi_q(\mathcal{E}_{k,\ell-1}^{(s)})\chi_q(\mathcal{E}_{k-1,\ell}^{(s+2)})$  are

$$\begin{aligned} M_0 &= M, \quad M_1 = M A_{2,s+2k+3\ell-3}^{-1}, \quad M_2 = M_1 A_{2,s+2k+3\ell-9}^{-1}, \quad \dots, \\ M_r &= M_{r-1} A_{2,s+2k+6}^{-1}, \quad M_{r+1} = M_r A_{1,s+2k-1}^{-1} A_{1,s+2k-3}^{-1} A_{2,s+2k}^{-1}, \\ M_{r+2} &= M_{r+1} A_{1,s+2k-5}^{-1}, \quad M_{r+3} = M_{r+2} A_{1,s+2k-7}^{-1}, \quad \dots, \quad M_{k+r-1} = M_{k+r-2} A_{1,s+1}^{-1}. \end{aligned}$$

The dominant monomials in  $\chi_q(\mathcal{E}_{k,\ell}^{(s)})\chi_q(\mathcal{E}_{k-1,\ell-1}^{(s+2)})$  are  $M_0, \dots, M_{k+r-2}$ .



If  $\ell = 2r$ , then dominant monomials in  $\chi_q(\mathcal{E}_{k,\ell-1}^{(s)})\chi_q(\mathcal{E}_{k-1,\ell}^{(s+2)})$  are

$$\begin{aligned} M_0 &= M, \quad M_1 = MA_{2,s+2k+3\ell-4}^{-1}, \quad M_2 = M_1A_{2,s+2k+3\ell-10}^{-1}, \quad \dots, \\ M_{r-1} &= M_{r-2}A_{2,s+2k+8}^{-1}, \quad M_r = M_{r-1}A_{1,s+2k-1}^{-1}A_{2,s+2k+2}^{-1}, \\ M_{r+1} &= M_rA_{1,s+2k-3}^{-1}, \quad M_{r+2} = M_{r+1}A_{1,s+2k-5}^{-1}, \quad \dots, \quad M_{k+r-1} = M_{k+r-2}A_{1,s+1}^{-1}. \end{aligned}$$

The dominant monomials in  $\chi_q(\mathcal{E}_{k,\ell}^{(s)})\chi_q(\mathcal{E}_{k-1,\ell-1}^{(s+2)})$  are  $M_0, \dots, M_{k+r-2}$ .

(6) Let  $M = F_{k,\ell-1}^{(s)}F_{k-1,\ell}^{(s+2)}$ ,  $k \geq 1, \ell \geq 1$ . Then dominant monomials in  $\chi_q(\mathcal{F}_{k,\ell-1}^{(s)})\chi_q(\mathcal{F}_{k-1,\ell}^{(s+2)})$  are

$$\begin{aligned} M_0 &= M, \quad M_1 = MA_{1,s+2k+2\ell+3}^{-1}, \quad M_2 = M_1A_{1,s+2k+2\ell+1}^{-1}, \quad \dots, \\ M_{\ell-1} &= M_{\ell-2}A_{1,s+2k+7}^{-1}, \quad M_\ell = M_{\ell-1}A_{1,s+2k-1}^{-1}A_{2,s+2k+2}^{-1}A_{1,s+2k+5}^{-1}, \\ M_{\ell+1} &= M_\ell A_{1,s+2k-3}^{-1}, \quad M_{\ell+2} = M_{\ell+1}A_{1,s+2k-5}^{-1}, \quad \dots, \quad M_{k+\ell-1} = M_{k+\ell-2}A_{1,s+1}^{-1}. \end{aligned}$$

The dominant monomials in  $\chi_q(\mathcal{F}_{k,\ell}^{(s)})\chi_q(\mathcal{F}_{k-1,\ell-1}^{(s+2)})$  are  $M_0, \dots, M_{k+\ell-2}$ .

In each case, for each  $i$ , the multiplicity of  $M_i$  in the corresponding product of  $q$ -characters is 1.

*Proof.* We prove the case of  $\chi_q(\mathcal{C}_{k,\ell-1}^{(s)})\chi_q(\mathcal{C}_{k-1,\ell}^{(s+6)})$ . The other cases are similar. Let  $m'_1 = C_{k,\ell-1}^{(s)}$ ,  $m'_2 = C_{k-1,\ell}^{(s+6)}$ . Without loss of generality, we assume that  $s = 6$ . Then

$$\begin{aligned} m'_1 &= (2_6 2_{12} \cdots 2_{6k})(2_{6k+10} 2_{6k+16} \cdots 2_{6k+6\ell-2}), \\ m'_2 &= (2_{12} \cdots 2_{6k})(2_{6k+10} 2_{6k+16} \cdots 2_{6k+6\ell-2} 2_{6k+6\ell+4}). \end{aligned}$$

Let  $m = m_1 m_2$  be a dominant monomial, where  $m_i \in \chi_q(m'_i)$ ,  $i = 1, 2$ . Denote by  $m_3 = 2_{6k+10} 2_{6k+16} \cdots 2_{6k+6\ell+4}$ . If  $m_2 \in \chi_q(2_{12} \cdots 2_{6k})(\chi_q(m_3) - m_3)$ , then  $m = m_1 m_2$  is right negative and hence  $m$  is not dominant. Therefore  $m_2 \in \chi_q(2_{12} \cdots 2_{6k})m_3$ .

Suppose that  $m_2 \in \mathcal{M}(L(m'_2)) \cap \mathcal{M}(\chi_q(2_{12} \cdots 2_{6k-6})(\chi_q(2_{6k}) - 2_{6k})m_3)$ . By the FM algorithm for  $L(m'_2)$  and Lemma 3.1,  $m_2$  must have a factor  $2_{6k+6}^{-1}$  or  $1_{6k+7}^{-1}$  or  $2_{6k+8}^{-1}$ . By Lemma 3.1,  $m_1$  does not have the factors  $2_{6k+6}$  and  $2_{6k+8}$ . Therefore  $m_2$  cannot have factors  $2_{6k+6}^{-1}$  and  $2_{6k+8}^{-1}$  since  $m = m_1 m_2$  is dominant. Hence  $1_{6k+7}^{-1}$  is a factor of  $m_2$ . Since  $m = m_1 m_2$  is dominant, we need to cancel  $1_{6k+7}^{-1}$  using a factor in  $m_1$ . By Lemma 3.1, the only possible way to cancel  $1_{6k+7}^{-1}$  by  $m_1$  is to use the factor  $1_{6k+5} 1_{6k+7} 1_{6k+9} 2_{6k+8}^{-1} 2_{6k+10}^{-1}$  or  $1_{6k+5} 1_{6k+7} 1_{6k+11}^{-1} 2_{6k+8}^{-1}$  of  $m_1$  coming from  $\chi_q(2_{6k})$ . Since  $2_{6k+8}^{-1}$  cannot be canceled by any monomials in  $\chi_q(2_6 2_{12} \cdots 2_{6k-6})$ , we have the factor  $2_{6k+8}^{-1}$  in  $m = m_1 m_2$  and hence  $m$  is not dominant. Therefore  $m_2 \in \mathcal{M}(L(2_{12} \cdots 2_{6k-6}))2_{6k} m_3$ . By the FM algorithm,  $m_2 = m'_2$ .

If

$$\begin{aligned} m_1 &\in \chi_q(2_6 \cdots 2_{6k} 2_{6k+10} 2_{6k+16} \cdots 2_{6k+6\ell-8})(\chi_q(2_{6k+6\ell-2}) \\ &\quad - 2_{6k+6\ell-2} - 2_{6k+6\ell+4}^{-1} 1_{6k+6\ell-1} 1_{6k+6\ell+1} 1_{6k+6\ell+3}), \end{aligned}$$

then  $m = m_1 m_2$  is right-negative and hence not dominant. Therefore  $m_1$  is in one of the following polynomials

$$\chi_q(2_6 \cdots 2_{6k} 2_{6k+10} 2_{6k+16} \cdots 2_{6k+6\ell-8}) 2_{6k+6\ell-2}, \quad (5.1)$$

$$\chi_q(2_6 \cdots 2_{6k} 2_{6k+10} 2_{6k+16} \cdots 2_{6k+6\ell-8}) 2_{6k+6\ell+4}^{-1} 1_{6k+6\ell-1} 1_{6k+6\ell+1} 1_{6k+6\ell+3}. \quad (5.2)$$

If  $m_1$  is in (5.1), then  $m_1 = m'_1$ . The dominant monomial we obtain is  $M_0 = m'_1 m'_2$ . If  $m_1$  is the highest monomial in (5.2), then we obtain the dominant monomial  $M_1 = m_1 m'_2$ . Suppose that  $m_1$  is in

$$\mathcal{M}(L(m'_1)) \cap \mathcal{M}(\chi_q(2_6 \cdots 2_{6k} 2_{6k+10} 2_{6k+16} \cdots 2_{6k+6\ell-14}) (\chi_q(2_{6k+6\ell-8}) - 2_{6k+6\ell-8}) 2_{6k+6\ell+4}^{-1} 1_{6k+6\ell-1} 1_{6k+6\ell+1} 1_{6k+6\ell+3}).$$

By the FM algorithm for  $L(m'_1)$ ,

$$m_1 \in \chi_q(2_6 \cdots 2_{6k} 2_{6k+10} 2_{6k+16} \cdots 2_{6k+6\ell-14}) \times (2_{6k+6\ell-2}^{-1} 1_{6k+6\ell-7} 1_{6k+6\ell-5} 1_{6k+6\ell-3}) (2_{6k+6\ell+4}^{-1} 1_{6k+6\ell-1} 1_{6k+6\ell+1} 1_{6k+6\ell+3}).$$

We obtain the dominant monomial  $M_2 = m_1 m'_2$ . Continue this procedure, we obtain dominant monomials  $M_3, \dots, M_{\ell-1}$  and the remaining dominant monomials are of the form  $m_1 m'_2$ , where  $m_1$  is a non-highest monomial in

$$\mathcal{M}(L(m'_1)) \cap \mathcal{M}(L(2_6 \cdots 2_{6k})) 2_{6k+16}^{-1} 2_{6k+22}^{-1} \cdots 2_{6k+6\ell+4}^{-1} 1_{6k+11} 1_{6k+13} \cdots 1_{6k+6\ell+3}.$$

Suppose that  $m_1$  is a non-highest monomial in the above set. Since the non-highest monomials in  $\chi_q(2_6 \cdots 2_{6k})$  are right-negative, we need cancellations of factors with negative powers of some monomial in  $\chi_q(2_6 \cdots 2_{6k})$  with  $2_{6k+10} 1_{6k+11} 1_{6k+13} \cdots 1_{6k+6\ell+3}$ . The only cancellation can happen is to cancel  $2_{6k+10}$  or  $1_{6k+11}$ . Since  $1_{6k+9}^2$  does not appear in  $\chi_q(2_6 \cdots 2_{6k})$ ,  $1_{6k+11}$  cannot be canceled. Therefore we need a cancellation with  $2_{6k+10}$ . The only monomials in  $\chi_q(2_6 \cdots 2_{6k})$  which can cancel  $2_{6k+10}$  is in one of the following polynomials

$$\begin{aligned} & \chi_q(2_6 \cdots 2_{6k-6}) 1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1}, \\ & \chi_q(2_6 \cdots 2_{6k-6}) 1_{6k+3}^{-1} 1_{6k+9} 2_{6k+2} 2_{6k+10}^{-1}, \\ & \chi_q(2_6 \cdots 2_{6k-6}) 1_{6k+5} 1_{6k+7} 1_{6k+9} 2_{6k+8}^{-1} 2_{6k+10}^{-1}. \end{aligned}$$

Therefore  $m_1$  is in one of the following sets

$$\mathcal{M}(L(m'_1)) \cap \mathcal{M}(L(2_6 \cdots 2_{6k-6})) 1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1} \cdots 2_{6k+6\ell+4}^{-1} 1_{6k+11} \cdots 1_{6k+6\ell+3}, \quad (5.3)$$

$$\mathcal{M}(L(m'_1)) \cap \mathcal{M}(L(2_6 \cdots 2_{6k-6})) 1_{6k+3}^{-1} 1_{6k+9} 2_{6k+2} 2_{6k+10}^{-1} \cdots 2_{6k+6\ell+4}^{-1} 1_{6k+11} \cdots 1_{6k+6\ell+3}, \quad (5.4)$$

$$\mathcal{M}(L(m'_1)) \cap \mathcal{M}(L(2_6 \cdots 2_{6k-6})) 1_{6k+5} 1_{6k+7} 1_{6k+9} 2_{6k+8}^{-1} 2_{6k+10}^{-1} \cdots 2_{6k+6\ell+4}^{-1} 1_{6k+11} \cdots 1_{6k+6\ell+3}. \quad (5.5)$$

If  $m_1$  is in (5.4), then we need to cancel  $1_{6k+3}^{-1}$ . We have

$$\mathcal{M}(L(2_6 \cdots 2_{6k-6})) \subset \mathcal{M}(\chi_q(2_6 \cdots 2_{6k-12}) \chi_q(2_{6k-6})).$$

By Lemma 3.1, only the monomials

$$1_{6k-5} 1_{6k+3} 2_{6k+4}^{-1}, \quad 1_{6k-3}^{-1} 1_{6k+3} 2_{6k-4} 2_{6k+4}^{-1}, \quad 1_{6k-1} 1_{6k+1} 1_{6k+3} 2_{6k+2}^{-1} 2_{6k+4}^{-1}$$

in  $\chi_q(2_{6k-6})$  can cancel  $1_{6k+3}^{-1}$ . But these monomials have the factor  $2_{6k+4}^{-1}$  which cannot be canceled by any monomials in  $\chi_q(2_6 \cdots 2_{6k-12})$  or by  $m'_2$ . Hence  $m_1$  is not in (5.4).

If  $m_1$  is in (5.5), then we need to cancel  $2_{6k+8}^{-1}$ . But  $2_{6k+8}^{-1}$  cannot be canceled by any monomials in  $\chi_q(2_6 \cdots 2_{6k-6})$  or by  $m'_2$ . Therefore  $m_1$  is not in (5.5). Hence  $m_1$  is in (5.3).

If  $m_1$  is the highest monomial in (5.3) with respect to  $\leq$  defined in (2.7), then  $m_1 m'_2 = M_\ell$ . Suppose that  $m_1$  is a non-highest monomial in (5.3). By the FM algorithm,  $m_1$  must in

$$\chi_q(2_6 \cdots 2_{6k-12}) 2_{6k}^{-1} 1_{6k-5} 1_{6k-3} 1_{6k-1} 1_{6k+1} 1_{6k+9} 2_{6k+10}^{-1} \cdots 2_{6k+6\ell+4}^{-1} 1_{6k+11} \cdots 1_{6k+6\ell+3}.$$

If  $m_1$  is the highest monomial in the above set, then  $m_1 m'_2 = M_{\ell+1}$ . Continue this procedure, we can show that the only remaining dominant monomials are  $M_{\ell+2}, \dots, M_{k+\ell-1}$ .

It is clear that the multiplicity of  $M_i, i = 1, \dots, k + \ell - 1$ , in  $\chi_q(m_1) \chi_q(m_2)$  is 1.  $\square$

## 5.2. Products of sources are special.

**Lemma 5.2.** *Let  $[S]$  be the last summand in one of the relations (3.4)–(3.11). Then  $S$  is special.*

*Proof.* We give a proof for  $S$  in the last line of (3.7) and in the last line of (3.11). The other cases are similar.

Let  $S_1 = \chi_q(\mathcal{C}_{r+1,p}^{(s+1)}) \chi_q(\mathcal{B}_{r+1,3p-1}^{(s+3)}) \chi_q(\mathcal{B}_{r+p+1,0}^{(s+5)})$ . Let

$$\begin{aligned} n_1 &= 2_{s+1} 2_{s+7} \cdots 2_{s+6r-5} 2_{s+6r+1}, \quad n'_1 = 2_{s+6r+11} 2_{s+6r+17} \cdots 2_{s+6r+6p+5}, \\ n_2 &= 2_{s+3} 2_{s+9} \cdots 2_{s+6r-3} 2_{s+6r+3}, \quad n'_2 = 1_{s+6r+10} 1_{s+6r+12} \cdots 1_{s+6r+6p+6}, \\ n_3 &= 2_{s+5} 2_{s+11} \cdots 2_{s+6r+6p+5}. \end{aligned}$$

Then  $C_{r+1,p}^{(s+1)} = n_1 n'_1, B_{r+1,3p-1}^{(s+3)} = n_2 n'_2, B_{r+p+1,0}^{(s+5)} = n_3$ . Let  $m' = m_1 m_2 m_3$  be a dominant monomial, where

$$m_1 \in \mathcal{M}(\mathcal{C}_{r+1,p}^{(s+1)}), \quad m_2 \in \mathcal{M}(\mathcal{B}_{r+1,3p-1}^{(s+3)}), \quad m_3 \in \mathcal{M}(\mathcal{B}_{r+p+1,0}^{(s+5)}).$$

If  $m_3 \neq B_{r+p+1,0}^{(s+5)}$  or  $m_1 \in \chi_q(n_1)(\chi_q(n'_1) - n'_1)$  or  $m_2 \in \chi_q(n_2)(\chi_q(n'_2) - n'_2)$ , then  $m'$  is right-negative which contradicts the fact that  $m'$  is dominant. Therefore  $m_3 = B_{r+p+1,0}^{(s+5)}, m_1 \in \chi_q(n_1) n'_1$ , and  $m_2 \in \chi_q(n_2) n'_2$ .

If  $m_2$  is in

$$\mathcal{M}(L(n_2 n'_2)) \cap \mathcal{M}(\chi_q(2_{s+3} 2_{s+9} \cdots 2_{s+6r-3})(\chi_q(2_{s+6r+3}) - 2_{s+6r+3}) n'_2), \quad (5.6)$$

then

$$m_2 \in \chi_q(2_{s+3} 2_{s+9} \cdots 2_{s+6r-3}) 2_{s+6r+9}^{-1} 1_{s+6k+4} 1_{s+6k+6} 1_{s+6k+8} n'_2.$$

By Lemma 3.1, the factor  $2_{s+6r+9}^{-1}$  cannot be canceled by any monomial in either  $\chi_q(n_1)$  or  $\chi_q(2_{s+3} 2_{s+9} \cdots 2_{s+6r-3})$ . It is clear that  $2_{s+6r+9}^{-1}$  cannot be canceled by  $n'_1, n'_2, n_3$ . Therefore  $2_{s+6r+9}^{-1}$  cannot be canceled. Hence  $m_2$  is not in (5.6). Thus  $m_2$  must be in

$$\mathcal{M}(L(n_2 n'_2)) \cap \mathcal{M}(L(2_{s+3} 2_{s+9} \cdots 2_{s+6r-3}) 2_{s+6r+3} n'_2).$$

Therefore  $m_2 = B_{r+1,3p-1}^{(s+3)}$ .

Suppose that  $m_1 \neq C_{r+1,p}^{(s+1)}$ . Then  $m_1 = m'_1 n'_1$ , where  $m'_1$  is a non-highest monomial in  $\chi_q(n_1)$ . Since the non-highest monomials in  $\chi_q(n_1)$  are right-negative, we need a cancellation with  $n'_1 n'_2 m_3$ . The only cancellation can happen is to cancel  $2_{s+6r+11}$  in  $n'_1$ , or cancel one of  $2_{s+6r+3}, 1_{s+6r+10}$  in  $n_2 n'_2$ , or cancel one of  $2_{s+6r+5}, 2_{s+6r+11}$  in  $m_3$ . By the FM algorithm,  $2_{s+6r+11}$  cannot be canceled. By Lemma 3.1,  $1_{s+6r+10}, 2_{s+6r+3}$  and  $2_{s+6r+5}$  cannot be canceled. This is a contradiction. Therefore  $m_1 = C_{r+1,p}^{(s+1)}$ .

Therefore the only dominant monomial in  $S_1$  is  $C_{r+1,p}^{(s+1)} B_{r+1,3p-1}^{(s+3)} B_{r+p+1,0}^{(s+5)}$ .

Let  $S_2 = \chi_q(\mathcal{D}_{r+1, \lfloor \frac{\ell}{3} \rfloor}^{(s+1)}) \chi_q(\mathcal{C}_{r+1, \lfloor \frac{\ell+1}{3} \rfloor}^{(s+3)}) \chi_q(\mathcal{B}_{r,0}^{(s+5)}) \chi_q(\mathcal{B}_{\lfloor \frac{\ell-1}{3} \rfloor, 0}^{(s+6r+17)})$ ,  $r \geq 0$ , and  $\ell = 3p, p \geq 1$ . The cases of  $\ell = 3p + 1, p \geq 0$  and  $\ell = 3p + 2, p \geq 0$  are similar. Let

$$\begin{aligned} n_1 &= 2_{s+1} 2_{s+7} \cdots 2_{s+6r+1} 1_{s+6r+8}, \quad n'_1 = 2_{s+6r+15} 2_{s+6r+21} \cdots 2_{s+6r+6p+9}, \\ n_2 &= 2_{s+3} 2_{s+9} \cdots 2_{s+6r-3} 2_{s+6r+3}, \quad n'_2 = 2_{s+6r+13} 2_{s+6r+20} \cdots 2_{s+6r+6p+7}, \\ n_3 &= 2_{s+5} 2_{s+11} \cdots 2_{s+6r-1}, \\ n_4 &= 2_{s+6r+17} 2_{s+6r+23} \cdots 2_{s+6r+6p+5}. \end{aligned}$$

Then  $D_{r+1, \lfloor \frac{\ell}{3} \rfloor}^{(s+1)} = n_1 n'_1$ ,  $C_{r+1, \lfloor \frac{\ell+1}{3} \rfloor}^{(s+3)} = n_2 n'_2$ ,  $B_{r,0}^{(s+5)} = n_3$ ,  $B_{\lfloor \frac{\ell-1}{3} \rfloor, 0}^{(s+6r+17)} = n_4$ .

Let  $m' = m_1 m_2 m_3 m_4$  be a dominant monomial, where

$$m_1 \in \mathcal{M}(\mathcal{D}_{r+1, \lfloor \frac{\ell}{3} \rfloor}^{(s+1)}), \quad m_2 \in \mathcal{M}(\mathcal{C}_{r+1, \lfloor \frac{\ell+1}{3} \rfloor}^{(s+3)}), \quad m_3 \in \mathcal{M}(\mathcal{B}_{r,0}^{(s+5)}), \quad m_4 \in \mathcal{M}(\mathcal{B}_{\lfloor \frac{\ell-1}{3} \rfloor, 0}^{(s+6r+17)}).$$

If  $m_4 \neq n_4$  or  $m_1 \in \chi_q(n_1)(\chi_q(n'_1) - n'_1)$  or  $m_2 \in \chi_q(n_2)(\chi_q(n'_2) - n'_2)$ , then  $m'$  is right-negative which contradicts the fact that  $m'$  is dominant. Therefore  $m_4 = n_4$ ,  $m_1 \in \chi_q(n_1)n'_1$ , and  $m_2 \in \chi_q(n_2)n'_2$ .

If

$$m_1 \in \mathcal{M}(L(n_1 n'_1)) \cap \mathcal{M}(\chi_q(2_{s+1} 2_{s+7} \cdots 2_{s+6r+1})(\chi_q(1_{s+6r+8}) - 1_{s+6r+8})n'_1), \quad (5.7)$$

then by the FM algorithm for  $L(n_1 n'_1)$ ,

$$m_1 \in \chi_q(2_{s+1} 2_{s+7} \cdots 2_{s+6r+1}) 1_{s+6r+10}^{-1} 2_{s+6r+9} n'_1.$$

It is clear that  $1_{s+6r+10}^{-1}$  is not canceled by  $n'_1, n'_2, n_4$ , and any monomial in  $\chi_q(n_3)$ . By the FM algorithm for  $\chi_q(n_2 n'_2)$ ,  $1_{s+6r+10}^{-1}$  cannot be canceled by any monomial in  $\chi_q(n_2 n'_2)$ . Therefore, by Lemma 3.1,  $1_{s+6r+10}^{-1}$  can only be canceled by one of the factors

$$\begin{aligned} &1_{s+6r+2} 1_{s+6r+10} 2_{s+6r+11}^{-1}, \\ &1_{s+6r+4}^{-1} 1_{s+6r+10} 2_{s+6r+3} 2_{s+6r+11}^{-1}, \quad 1_{s+6r+6} 1_{s+6r+8} 1_{s+6r+10} 2_{s+6r+9}^{-1} 2_{s+6r+11}^{-1} \end{aligned}$$

coming from  $\chi_q(2_{s+6r+1})$ , where  $2_{s+6r+1}$  is in  $n_1$ . But then  $2_{s+6r+11}^{-1}$  cannot be canceled. This contradicts the fact that  $m'$  is dominant. Hence  $m_1$  is not in (5.6). Thus  $m_1$  must be in

$$\mathcal{M}(L(n_1 n'_1)) \cap \mathcal{M}(L(n_1 n'_1)) \cap \mathcal{M}(L(2_{s+1} 2_{s+7} \cdots 2_{s+6r+1}) 1_{s+6r+8} n'_1).$$

If  $m_1$  is in

$$\begin{aligned} &\mathcal{M}(L(n_1 n'_1)) \cap \mathcal{M}(L(n_1 n'_1)) \cap \mathcal{M}(\chi_q(2_{s+1} \times \\ &\quad \times 2_{s+7} \cdots 2_{s+6r-5})(\chi_q(2_{s+6r+1}) - 2_{s+6r+1}) 1_{s+6r+8} n'_1). \end{aligned}$$

Then

$$m_1 \in \chi_q(2_{s+1} 2_{s+7} \cdots 2_{s+6r-5}) 2_{s+6r+7}^{-1} 1_{s+6r+2} 1_{s+6r+4} 1_{s+6r+6} 1_{s+6r+8} n'_1.$$

The only possible way to cancel  $2_{s+6r+7}^{-1}$  is to use one of the terms

$$1_{s+6r+4} 1_{s+6r+8}^{-1} 1_{s+6r+10}^{-1} 2_{s+6r+7}, \quad 1_{s+6r+6}^{-1} 1_{s+6r+8}^{-1} 1_{s+6r+10}^{-1} 2_{s+6r+5} 2_{s+6r+7}, \quad 2_{s+6r+7} 2_{s+6r+11}^{-1} \quad (5.8)$$

in  $\chi_q(2_{s+6r+3})$ , where  $2_{s+6r+3}$  is in  $n_2$ . But then we have to cancel  $1_{s+6r+10}^{-1}$  or  $2_{s+6r+11}^{-1}$ . But  $1_{s+6r+10}^{-1}$  and  $2_{s+6r+11}^{-1}$  cannot be canceled. This is a contradiction. Therefore  $m_1$  must in

$$\mathcal{M}(L(n_1 n'_1)) \cap \mathcal{M}(L(2_{s+1} 2_{s+7} \cdots 2_{s+6r-5})) 2_{s+6r+1} 1_{s+6r+8} n'_1.$$

Hence  $m_1 = n_1 n'_1$ .

By the FM algorithm, when we compute the  $q$ -character for  $\chi_q(n_2 n'_2)$ , we can only choose one of the following terms

$$2_{s+6r+3}, 1_{s+6r+4} 1_{s+6r+6} 1_{s+6r+8} 2_{s+6r+9}^{-1}, 1_{s+6r+4} 1_{s+6r+6} 1_{s+6r+10}^{-1}, \\ 1_{s+6r+4} 1_{s+6r+8}^{-1} 1_{s+6r+10}^{-1} 2_{s+6r+7}, 1_{s+6r+6}^{-1} 1_{s+6r+8}^{-1} 1_{s+6r+10}^{-1} 2_{s+6r+5} 2_{s+6r+7}, 2_{s+6r+11}^{-1} 2_{s+6r+7}$$

in  $\chi_q(2_{s+6r+3})$ . Since  $2_{s+6r+9}^{-1}$ ,  $1_{s+6r+10}^{-1}$ , and  $2_{s+6r+11}^{-1}$  cannot be canceled, we can only choose  $2_{s+6r+3}$ . Therefore  $m_2$  is in

$$\mathcal{M}(L(n_2 n'_2)) \cap \mathcal{M}(L(2_{s+3} 2_{s+9} \cdots 2_{s+6r-3})) 2_{s+6r+3} n'_2.$$

Therefore  $m_2 = n_2 n'_2$ .

If  $m_3$  is in

$$\mathcal{M}(L(n_3)) \cap \mathcal{M}(\chi_q(2_{s+5} 2_{s+11} \cdots 2_{s+6r-7})(\chi_q(2_{s+6r-1}) - 2_{s+6r-1})),$$

then, by Lemma 3.1,  $m = m_1 m_2 m_3 m_4$  is non-dominant since  $m_1 = n_1 n'_1$ ,  $m_2 = n_2 n'_2$ ,  $m_4 = n_4$ . This contradicts the fact that  $m$  is dominant. Therefore  $m_3$  is in

$$\mathcal{M}(L(n_3)) \cap \mathcal{M}(L(2_{s+5} 2_{s+11} \cdots 2_{s+6r-7})) 2_{s+6r-1}.$$

Hence  $m_3 = n_3$ .

Therefore the only dominant monomial in  $S_2$  is  $D_{r+1, \lfloor \frac{\ell}{3} \rfloor}^{(s+1)} C_{r+1, \lfloor \frac{\ell+1}{3} \rfloor}^{(s+3)} B_{r,0}^{(s+5)} B_{\lfloor \frac{\ell-1}{3} \rfloor, 0}^{(s+6r+17)}$ .  $\square$

**5.3. Proof of Theorem 3.4.** By Lemmas 5.1 and 5.2, the dominant monomials in the  $q$ -characters of the left hand side and of the right hand side of every relation in Theorem 3.4 are the same. The theorem follows.

## 6. PROOF OF THEOREM 3.5

By Lemma 5.2,  $\mathcal{S}$  is special and hence irreducible. Therefore we only have to show that  $\mathcal{T} \otimes \mathcal{B}$  is irreducible. It suffices to prove that for each non-highest dominant monomial  $M$  in  $\mathcal{T} \otimes \mathcal{B}$ , we have  $\mathcal{M}(L(M)) \not\subset \mathcal{M}(\mathcal{T} \otimes \mathcal{B})$ . The idea is similar as in [Her06], [MY11b]. Recall that the dominant monomials in  $\mathcal{T} \otimes \mathcal{B}$  are described by Lemma 5.1.

**Lemma 6.1.** *We consider the same cases as in Lemma 5.1. In each case  $M_i$  are the dominant monomials described by that lemma.*

(1) For  $k \geq 1, \ell \geq 1$ , let

$$n_1 = M_1 A_{1,s+6k+2\ell-2}^{-1}, \quad n_2 = M_2 A_{1,s+6k+2\ell-4}^{-1}, \quad \dots, \\ n_{\ell-1} = M_{\ell-1} A_{1,s+6k+2}^{-1}, \quad n_\ell = M_\ell A_{2,s+6k-3}^{-1} A_{1,s+6k}^{-1}, \\ n_{\ell+1} = M_{\ell+1} A_{2,s+6k-9}^{-1}, \quad \dots, \quad n_{k+\ell-2} = M_{k+\ell-2} A_{2,s+9}^{-1}.$$

Then for  $i = 1, \dots, k + \ell - 2$ ,  $n_i \in \chi_q(M_i)$  and  $n_i \notin \chi_q(\mathcal{B}_{k,\ell}^{(s)}) \chi_q(\mathcal{B}_{k-1,\ell-1}^{(s+6)})$ .

(2) For  $k \geq 1, \ell \geq 1$ , let

$$\begin{aligned} n_1 &= M_1 A_{2,s+6k+6\ell-5}^{-1}, \quad n_2 = M_2 A_{2,s+6k+6\ell-11}^{-1}, \quad \dots, \\ n_{\ell-1} &= M_{\ell-1} A_{2,s+6k+7}^{-1}, \quad n_\ell = M_\ell A_{2,s+6k-3}^{-1} A_{1,s+6k}^{-1} A_{1,s+6k-2}^{-1} A_{2,s+6k+1}^{-1}, \\ n_{\ell+1} &= M_{\ell+1} A_{2,s+6k-9}^{-1}, \quad \dots, \quad n_{k+\ell-2} = M_{k+\ell-2} A_{2,s+9}^{-1}. \end{aligned}$$

Then for  $i = 1, \dots, k + \ell - 2$ ,  $n_i \in \chi_q(M_i)$  and  $n_i \notin \chi_q(\mathcal{C}_{k,\ell}^{(s)}) \chi_q(\mathcal{C}_{k-1,\ell-1}^{(s+6)})$ .

(3) For  $k \geq 0, \ell \geq 1$ , let

$$\begin{aligned} n_1 &= M_1 A_{2,s+6k+6\ell-1}^{-1}, \quad n_2 = M_2 A_{2,s+6k+6\ell-7}^{-1}, \quad \dots, \\ n_{\ell-1} &= M_{\ell-1} A_{2,s+6k+11}^{-1}, \quad n_\ell = M_\ell A_{1,s+6k+2}^{-1} A_{2,s+6k+5}^{-1}, \\ n_{\ell+1} &= M_{\ell+1} A_{2,s+6k-3}^{-1} A_{1,s+6k}^{-1}, \quad n_{\ell+2} = M_{\ell+2} A_{2,s+6k-9}^{-1}, \quad \dots, \quad n_{k+\ell-1} = M_{k+\ell-1} A_{2,s+9}^{-1}. \end{aligned}$$

Then for  $i = 1, \dots, k + \ell - 1$ ,  $n_i \in \chi_q(M_i)$  and  $n_i \notin \chi_q(\mathcal{D}_{k-1,\ell-1}^{(s+6)}) \chi_q(\mathcal{D}_{k,\ell}^{(s)})$ .

(4) For  $k \geq 0, \ell = 2r + 1, r \geq 0$ , let

$$\begin{aligned} n_1 &= M_1 A_{2,s+2k+3\ell-3}^{-1}, \quad n_2 = M_2 A_{2,s+2k+3\ell-9}^{-1}, \quad \dots, \\ n_r &= M_r A_{2,s+2k+3}^{-1}, \quad n_{r+1} = M_{r+1} A_{1,s+2k-1}^{-1} A_{1,s+2k-3}^{-1} A_{2,s+2k}^{-1}, \\ n_{r+2} &= M_{r+2} A_{1,s+2k-5}^{-1}, \quad \dots, \quad n_{k+r-2} = M_{k+r-2} A_{1,s+3}^{-1}. \end{aligned}$$

Then for  $i = 1, \dots, r + k - 2$ ,  $n_i \in \chi_q(M_i)$  and  $n_i \notin \chi_q(\mathcal{E}_{k,\ell}^{(s)}) \chi_q(\mathcal{E}_{k-1,\ell-1}^{(s+2)})$ .

For  $k \geq 0, \ell = 2r, r \geq 1$ , let

$$\begin{aligned} n_1 &= M_1 A_{2,s+2k+3\ell-4}^{-1}, \quad n_2 = M_2 A_{2,s+2k+3\ell-10}^{-1}, \quad \dots, \\ n_{r-1} &= M_{r-1} A_{2,s+2k+8}^{-1}, \quad n_r = M_r A_{1,s+2k-1}^{-1} A_{2,s+2k+2}^{-1}, \\ n_{r+1} &= M_{r+1} A_{1,s+2k-3}^{-1}, \quad \dots, \quad n_{k+r-2} = M_{k+r-2} A_{1,s+3}^{-1}. \end{aligned}$$

Then for  $i = 1, \dots, r + k - 2$ ,  $n_i \in \chi_q(M_i)$  and  $n_i \notin \chi_q(\mathcal{E}_{k,\ell}^{(s)}) \chi_q(\mathcal{E}_{k-1,\ell-1}^{(s+2)})$ .

(5) For  $k \geq 1, \ell \geq 1$ , let

$$\begin{aligned} n_1 &= M_1 A_{1,s+2k+2\ell+3}^{-1}, \quad n_2 = M_2 A_{1,s+2k+2\ell+1}^{-1}, \quad \dots, \\ n_{\ell-1} &= M_{\ell-1} A_{1,s+2k+7}^{-1}, \quad n_\ell = M_\ell A_{1,s+2k-1}^{-1} A_{2,s+2k+2}^{-1} A_{1,s+2k+5}^{-1}, \\ n_{\ell+1} &= M_{\ell+1} A_{1,s+2k-3}^{-1}, \quad \dots, \quad n_{k+\ell-2} = M_{k+\ell-2} A_{1,s+3}^{-1}. \end{aligned}$$

Then  $i = 1, \dots, k + \ell - 2$ ,  $n_i \in \chi_q(M_i)$  and  $n_i \notin \chi_q(\mathcal{F}_{k,\ell}^{(s)}) \chi_q(\mathcal{F}_{k-1,\ell-1}^{(s+2)})$ .

*Proof.* We give a proof in the case of  $\chi_q(\mathcal{C}_{k,\ell}^{(s)}) \chi_q(\mathcal{C}_{k-1,\ell-1}^{(s+6)})$ . The other cases are similar. By definition, we have

$$\begin{aligned} C_{k,\ell}^{(s)} &= (2_s 2_{s+6} \cdots 2_{s+6k-6}) (2_{s+6k+4} 2_{s+6k+10} \cdots 2_{s+6k+6\ell-8} 2_{s+6k+6\ell-2}), \\ C_{k-1,\ell-1}^{(s+6)} &= (2_{s+6} 2_{s+12} \cdots 2_{s+6k-6}) (2_{s+6k+4} 2_{s+6k+10} \cdots 2_{s+6k+6\ell-8}), \\ M_1 &= C_{k,\ell}^{(s)} C_{k-1,\ell-1}^{(s+6)} A_{2,s+6k+6\ell-5}^{-1} \\ &= C_{k,\ell}^{(s)} C_{k-1,\ell-1}^{(s+6)} 2_{s+6k+6\ell-8}^{-1} 2_{s+6k+6\ell-2}^{-1} 1_{s+6k+6\ell-7} 1_{s+6k+6\ell-5} 1_{s+6k+6\ell-3}. \end{aligned}$$

By  $U_{q_2}(\hat{\mathfrak{sl}}_2)$  argument, it is clear that  $n_1 = M_1 A_{2,s+6k+6\ell-5}^{-1}$  is in  $\chi_q(M_1)$ .

If  $n_1$  is in  $\chi_q(\mathcal{C}_{k,\ell}^{(s)})\chi_q(\mathcal{C}_{k-1,\ell-1}^{(s+6)})$ , then  $C_{k,\ell}^{(s)} A_{2,s+6k+6\ell-5}^{-1}$  is in  $\chi_q(\mathcal{C}_{k,\ell}^{(s)})$  which is impossible by the FM algorithm for  $\mathcal{C}_{k,\ell}^{(s)}$ . Similarly,  $n_i \in \chi_q(M_i)$ ,  $i = 2, \dots, \ell - 1$ , but  $n_2, \dots, n_{\ell-1}$  are not in  $\chi_q(\mathcal{C}_{k,\ell}^{(s)})\chi_q(\mathcal{C}_{k-1,\ell-1}^{(s+6)})$ .

By definition,

$$M_\ell = (2_s 2_{s+6} \cdots 2_{s+6k-6})(2_{s+6} 2_{s+12} \cdots 2_{s+6k-12})(1_{s+6k-5} 1_{s+6k+3} 1_{s+6k+5} \cdots 1_{s+6k+6\ell-3}).$$

Let  $U = \{(1, aq^{s+6k}), (1, aq^{s+6k-3}), (2, aq^{s+6k-2}), (2, aq^{s+6k+1})\} \subset I \times \mathbb{C}^\times$ . Let  $\mathcal{M}$  be the finite set consisting of the following monomials

$$m_0 = M_\ell, \quad m_1 = m_0 A_{2,s+6k-3}^{-1}, \quad m_2 = m_1 A_{1,s+6k}^{-1}, \quad m_3 = m_2 A_{1,s+6k-2}^{-1}, \quad m_4 = m_3 A_{2,s+6k+1}^{-1}.$$

It is clear that  $\mathcal{M}$  satisfies the conditions in Theorem 2.2. Therefore

$$\text{trunc}_{m+\mathcal{Q}_U^-}(\chi_q(M_\ell)) = \sum_{m \in \mathcal{M}} m$$

and hence  $n_\ell = M_\ell A_{2,s+6k-3}^{-1} A_{1,s+6k}^{-1} A_{1,s+6k-2}^{-1} A_{2,s+6k+1}^{-1}$  is in  $\chi_q(M_\ell)$ .

If  $n_\ell$  is in  $\chi_q(\mathcal{C}_{k,\ell}^{(s)})\chi_q(\mathcal{C}_{k-1,\ell-1}^{(s+6)})$ , then  $C_{k,\ell}^{(s)} A_{2,s+6k-3}^{-1} A_{1,s+6k}^{-1} A_{1,s+6k-2}^{-1} A_{2,s+6k+1}^{-1}$  is in  $\chi_q(\mathcal{C}_{k,\ell}^{(s)})$  which is impossible by the FM algorithm for  $\mathcal{C}_{k,\ell}^{(s)}$ .

Similarly, we show that for  $i = \ell + 1, \dots, k + \ell - 2$ ,  $n_i \in \chi_q(M_i)$  and  $n_i \notin \chi_q(\mathcal{C}_{k,\ell}^{(s)})\chi_q(\mathcal{C}_{k-1,\ell-1}^{(s+6)})$ .  $\square$

## 7. THE SECOND PART OF THE EXTENDED T-SYSTEM

Let  $\tilde{B}_{k,\ell}^{(s)}, \tilde{C}_{k,\ell}^{(s)}, \tilde{D}_{k,\ell}^{(s)}, \tilde{E}_{k,\ell}^{(s)}, \tilde{F}_{k,\ell}^{(s)}$  be the monomials obtained from  $B_{k,\ell}^{(s)}, C_{k,\ell}^{(s)}, D_{k,\ell}^{(s)}, E_{k,\ell}^{(s)}, F_{k,\ell}^{(s)}$  by replacing  $i_a$  with  $i_{-a}$ ,  $i = 1, 2$ . Namely,

$$\begin{aligned} \tilde{B}_{k,\ell}^{(s)} &= \left( \prod_{i=0}^{\ell-1} 1_{-s-6k-2i-1} \right) \left( \prod_{i=0}^{k-1} 2_{-s-6i} \right), \quad \tilde{C}_{k,\ell}^{(s)} = \left( \prod_{i=0}^{\ell-1} 2_{-s-6k-6i-4} \right) \left( \prod_{i=0}^{k-1} 2_{-s-6i} \right), \\ \tilde{D}_{k,\ell}^{(s)} &= \left( \prod_{i=0}^{\ell-1} 2_{-s-6k-6i-8} \right) 1_{-s-6k-1} \left( \prod_{i=0}^{k-1} 2_{-s-6i} \right), \quad \tilde{F}_{k,\ell}^{(s)} = \left( \prod_{i=0}^{\ell-1} 1_{-s-2k-2i-6} \right) \left( \prod_{i=0}^{k-1} 1_{-s-2i} \right), \\ \tilde{E}_{k,\ell}^{(s)} &= \left( \prod_{i=0}^{\lfloor \frac{\ell-2}{2} \rfloor} 2_{-s-2k-6i-5} \right) \left( \prod_{i=0}^{\lfloor \frac{\ell-1}{2} \rfloor} 2_{-s-2k-6i-3} \right) \left( \prod_{i=0}^{k-1} 1_{-s-2i} \right). \end{aligned}$$

Note that, in particular, for  $k \in \mathbb{Z}_{\geq 0}$ ,  $s \in \mathbb{Z}$ , we have the following trivial relations

$$\tilde{\mathcal{B}}_{k,0}^{(s)} = \tilde{\mathcal{C}}_{k,0}^{(s)} = \tilde{\mathcal{C}}_{0,k}^{(s-4)}, \quad \tilde{\mathcal{D}}_{k,0}^{(s)} = \tilde{\mathcal{B}}_{k,1}^{(s)}, \quad \tilde{\mathcal{E}}_{k,0}^{(s)} = \tilde{\mathcal{B}}_{0,k}^{(s-1)} = \tilde{\mathcal{F}}_{0,k}^{(s-6)} = \tilde{\mathcal{F}}_{k,0}^{(s)}. \quad (7.1)$$

We also have  $\mathcal{D}_{0,k}^{(s)} = \tilde{\mathcal{B}}_{k,1}^{(-s-6k-2)}$ ,  $\tilde{\mathcal{D}}_{0,k}^{(s)} = \mathcal{B}_{k,1}^{(-s-6k-2)}$ ,  $k \in \mathbb{Z}_{\geq 0}$ ,  $s \in \mathbb{Z}$ .

Note that  $\tilde{\mathcal{B}}_{k,\ell}^{(s)}, \tilde{\mathcal{D}}_{0,\ell}^{(s)}, \tilde{\mathcal{D}}_{k,0}^{(s)}$  are minimal affinizations. In general, the modules  $\tilde{\mathcal{B}}_{k,\ell}^{(s)}, \tilde{\mathcal{C}}_{k,\ell}^{(s)}, \tilde{\mathcal{D}}_{k,\ell}^{(s)}, \tilde{\mathcal{E}}_{k,\ell}^{(s)}, \tilde{\mathcal{F}}_{k,\ell}^{(s)}$  are not special. For example, we have the following proposition.

**Proposition 7.1.** *The module  $\tilde{\mathcal{B}}_{3,1}^{(0)} = L(1_0 1_2 1_4 2_{11})$  is not special.*

*Proof.* Suppose that  $L(1_0 1_2 1_4 2_{11})$  is special. Then the FM algorithm applies to  $L(1_0 1_2 1_4 2_{11})$ . Therefore, by the FM algorithm, the monomials

$$1_0 1_2 1_4 2_{11}, 1_0 1_2 1_6^{-1} 2_5 2_{11}, 1_0 1_4^{-1} 1_6^{-1} 2_3 2_5 2_{11}, 1_2^{-1} 1_4^{-1} 1_6^{-1} 2_1 2_3 2_5 2_{11}, 2_7^{-1} 2_3 2_5 2_{11}, \\ 2_7^{-1} 2_9^{-1} 1_4 1_6 1_8 2_5 2_{11}, 2_7^{-1} 1_4 1_6 1_{10}^{-1} 2_5 2_{11}, 1_4 1_8^{-1} 1_{10}^{-1} 2_5 2_{11}, 1_6^{-1} 1_8^{-1} 1_{10}^{-1} 2_5^2 2_{11}, 2_5$$

are in  $\mathcal{M}(L(1_0 1_2 1_4 2_{11}))$ . Hence  $\mathcal{M}(L(1_0 1_2 1_4 2_{11}))$  has at least two dominant monomials  $1_0 1_2 1_4 2_{11}$  and  $2_5$ . This contradicts the assumption that  $L(1_0 1_2 1_4 2_{11})$  is special.  $\square$

**Theorem 7.2.** *The modules  $\tilde{\mathcal{B}}_{k,\ell}^{(s)}, \tilde{\mathcal{C}}_{k,\ell}^{(s)}, \tilde{\mathcal{D}}_{k,\ell}^{(s)}, \tilde{\mathcal{E}}_{k,\ell}^{(s)}, \tilde{\mathcal{F}}_{k,\ell}^{(s)}$ ,  $s \in \mathbb{Z}$ ,  $k, \ell \in \mathbb{Z}_{\geq 0}$  are anti-special.*

*Proof.* This theorem can be proved using the dual arguments of the proof of Theorem 3.3.  $\square$

**Lemma 7.3.** *Let  $\iota : \mathbb{ZP} \rightarrow \mathbb{ZP}$  be a homomorphism of rings such that  $Y_{1,aq^s} \mapsto Y_{1,aq^{12-s}}^{-1}$ ,  $Y_{2,aq^s} \mapsto Y_{2,aq^{12-s}}^{-1}$  for all  $a \in \mathbb{C}^\times$ ,  $s \in \mathbb{Z}$ . Then*

$$\chi_q(\tilde{\mathcal{B}}_{k,\ell}^{(s)}) = \iota(\chi_q(\mathcal{B}_{k,\ell}^{(s)})), \quad \chi_q(\tilde{\mathcal{C}}_{k,\ell}^{(s)}) = \iota(\chi_q(\mathcal{C}_{k,\ell}^{(s)})), \\ \chi_q(\tilde{\mathcal{D}}_{k,\ell}^{(s)}) = \iota(\chi_q(\mathcal{D}_{k,\ell}^{(s)})), \quad \chi_q(\tilde{\mathcal{E}}_{k,\ell}^{(s)}) = \iota(\chi_q(\mathcal{E}_{k,\ell}^{(s)})), \quad \chi_q(\tilde{\mathcal{F}}_{k,\ell}^{(s)}) = \iota(\chi_q(\mathcal{F}_{k,\ell}^{(s)})).$$

*Proof.* Let  $m_+$  be one of  $B_{k,\ell}^{(s)}, C_{k,\ell}^{(s)}, D_{k,\ell}^{(s)}, E_{k,\ell}^{(s)}, F_{k,\ell}^{(s)}$ . Then  $\chi_q(m_+)$  can be computed by the FM algorithm starting from the lowest weight using  $A_{i,a}$  with  $i \in I, a \in \mathbb{C}^\times$ . The procedure is dual to the computation of  $\chi_q(m_+)$  which starts from  $m_+$  using  $A_{i,a}^{-1}$  with  $i \in I, a \in \mathbb{C}^\times$ . The highest (resp. lowest)  $l$ -weight in  $\chi_q(m_+)$  is sent to the lowest (resp. highest)  $l$ -weight in  $\chi_q(\tilde{m}_+)$  by  $\iota$ .  $\square$

Note that Lemma 7.3 can also be proved using the Cartan involution in [CP91].

The modules  $\tilde{\mathcal{B}}_{k,\ell}^{(s)}, \tilde{\mathcal{C}}_{k,\ell}^{(s)}, \tilde{\mathcal{D}}_{k,\ell}^{(s)}, \tilde{\mathcal{E}}_{k,\ell}^{(s)}, \tilde{\mathcal{F}}_{k,\ell}^{(s)}$  satisfy the same relations as in Theorem 3.4 but the roles of left and right modules are exchanged. More precisely, we have the following theorem.

**Theorem 7.4.** *For  $s \in \mathbb{Z}$ ,  $k, \ell \in \mathbb{Z}_{\geq 1}$ ,  $t \in \mathbb{Z}_{\geq 2}$ , we have the following relations in  $\text{Rep}(U_q \hat{\mathfrak{g}})$ .*

$$[\tilde{\mathcal{B}}_{k-1,\ell}^{(s+6)}][\tilde{\mathcal{B}}_{k,\ell-1}^{(s)}] = [\tilde{\mathcal{B}}_{k,\ell}^{(s)}][\tilde{\mathcal{B}}_{k-1,\ell-1}^{(s+6)}] + [\tilde{\mathcal{E}}_{3k-1,\lceil \frac{2\ell-2}{3} \rceil}^{(s+1)}][\tilde{\mathcal{B}}_{\lfloor \frac{\ell-1}{3} \rfloor,0}^{(s+6k+6)}], \\ [\tilde{\mathcal{E}}_{0,\ell}^{(s)}] = [\tilde{\mathcal{B}}_{\lfloor \frac{\ell+1}{2} \rfloor,0}^{(s+3)}][\tilde{\mathcal{B}}_{\lfloor \frac{\ell}{2} \rfloor,0}^{(s+5)}], \\ [\tilde{\mathcal{E}}_{1,\ell}^{(s)}] = [\tilde{\mathcal{D}}_{0,\lfloor \frac{\ell}{2} \rfloor}^{(s-1)}][\tilde{\mathcal{B}}_{\lfloor \frac{\ell+1}{2} \rfloor,0}^{(s+5)}], \\ [\tilde{\mathcal{E}}_{t-1,\ell}^{(s+2)}][\tilde{\mathcal{E}}_{t,\ell-1}^{(s)}] = [\tilde{\mathcal{E}}_{t,\ell}^{(s)}][\tilde{\mathcal{E}}_{t-1,\ell-1}^{(s+2)}] + \begin{cases} [\tilde{\mathcal{D}}_{r,p-1}^{(s+1)}][\tilde{\mathcal{B}}_{r+p,0}^{(s+3)}][\tilde{\mathcal{B}}_{r,3p-2}^{(s+5)}] & \text{if } t = 3r + 2, \ell = 2p - 1, \\ [\tilde{\mathcal{B}}_{r+p+1,0}^{(s+1)}][\tilde{\mathcal{C}}_{r,p}^{(s+3)}][\tilde{\mathcal{B}}_{r,3p-1}^{(s+5)}] & \text{if } t = 3r + 2, \ell = 2p, \\ [\tilde{\mathcal{B}}_{r+1,3p-2}^{(s+1)}][\tilde{\mathcal{D}}_{r,p-1}^{(s+3)}][\tilde{\mathcal{B}}_{r+p,0}^{(s+5)}] & \text{if } t = 3r + 3, \ell = 2p - 1, \\ [\tilde{\mathcal{B}}_{r+1,3p-1}^{(s+1)}][\tilde{\mathcal{B}}_{r+p+1,0}^{(s+3)}][\tilde{\mathcal{C}}_{r,p}^{(s+5)}] & \text{if } t = 3r + 3, \ell = 2p, \\ [\tilde{\mathcal{B}}_{r+p+1,0}^{(s+1)}][\tilde{\mathcal{B}}_{r+1,3p-2}^{(s+3)}][\tilde{\mathcal{D}}_{r,p-1}^{(s+5)}] & \text{if } t = 3r + 4, \ell = 2p - 1, \\ [\tilde{\mathcal{C}}_{r+1,p}^{(s+1)}][\tilde{\mathcal{B}}_{r+1,3p-1}^{(s+3)}][\tilde{\mathcal{B}}_{r+p+1,0}^{(s+5)}] & \text{if } t = 3r + 4, \ell = 2p, \end{cases} \\ [\tilde{\mathcal{C}}_{k-1,\ell}^{(s+6)}][\tilde{\mathcal{C}}_{k,\ell-1}^{(s)}] = [\tilde{\mathcal{C}}_{k,\ell}^{(s)}][\tilde{\mathcal{C}}_{k-1,\ell-1}^{(s+6)}] + [\tilde{\mathcal{F}}_{3k-2,3\ell-2}^{(s+1)}],$$



$$[\tilde{\mathcal{B}}_{\ell,0}^{(s+8)}][\tilde{\mathcal{D}}_{0,\ell-1}^{(s)}] = [\tilde{\mathcal{D}}_{0,\ell}^{(s)}][\tilde{\mathcal{B}}_{\ell-1,0}^{(s+8)}] + [\tilde{\mathcal{B}}_{0,3\ell-1}^{(s+4)}],$$

$$[\tilde{\mathcal{D}}_{k-1,\ell}^{(s+6)}][\tilde{\mathcal{D}}_{k,\ell-1}^{(s)}] = [\tilde{\mathcal{D}}_{k,\ell}^{(s)}][\tilde{\mathcal{D}}_{k-1,\ell-1}^{(s+6)}] + [\tilde{\mathcal{F}}_{3k-1,3\ell-1}^{(s+1)}],$$

$$[\tilde{\mathcal{F}}_{k-1,\ell}^{(s+2)}][\tilde{\mathcal{F}}_{k,\ell-1}^{(s)}] = [\tilde{\mathcal{F}}_{k,\ell}^{(s)}][\tilde{\mathcal{F}}_{k-1,\ell-1}^{(s+2)}] + \begin{cases} [\tilde{\mathcal{B}}_{r,0}^{(s+1)}][\tilde{\mathcal{D}}_{r,\lfloor \frac{\ell}{3} \rfloor}^{(s+3)}][\tilde{\mathcal{C}}_{r,\lfloor \frac{\ell+1}{3} \rfloor}^{(s+5)}][\tilde{\mathcal{B}}_{\lfloor \frac{\ell-1}{3} \rfloor,0}^{(s+2k+11)}] & \text{if } k = 3r + 1, \\ [\tilde{\mathcal{C}}_{r+1,\lfloor \frac{\ell+1}{3} \rfloor}^{(s+1)}][\tilde{\mathcal{B}}_{r,0}^{(s+3)}][\tilde{\mathcal{D}}_{r,\lfloor \frac{\ell}{3} \rfloor}^{(s+5)}][\tilde{\mathcal{B}}_{\lfloor \frac{\ell-1}{3} \rfloor,0}^{(s+2k+11)}] & \text{if } k = 3r + 2, \\ [\tilde{\mathcal{D}}_{r+1,\lfloor \frac{\ell}{3} \rfloor}^{(s+1)}][\tilde{\mathcal{C}}_{r+1,\lfloor \frac{\ell+1}{3} \rfloor}^{(s+3)}][\tilde{\mathcal{B}}_{r,0}^{(s+5)}][\tilde{\mathcal{B}}_{\lfloor \frac{\ell-1}{3} \rfloor,0}^{(s+2k+11)}] & \text{if } k = 3r + 3. \end{cases}$$

Moreover, the modules corresponding to each summand on the right hand side of the above relations are all irreducible.

*Proof.* The theorem follows from the relations in Theorem 3.4, Theorem 3.5, and Lemma 7.3.  $\square$

The following proposition is similar to Proposition 3.6.

**Proposition 7.5.** *Given  $\chi_q(1_s), \chi_q(2_s)$ , one can obtain the  $q$ -characters of  $\tilde{\mathcal{B}}_{k,\ell}^{(s)}, \tilde{\mathcal{C}}_{k,\ell}^{(s)}, \tilde{\mathcal{D}}_{k,\ell}^{(s)}, \tilde{\mathcal{E}}_{k,\ell}^{(s)}, \tilde{\mathcal{F}}_{k,\ell}^{(s)}$ ,  $s \in \mathbb{Z}$ ,  $k, \ell \in \mathbb{Z}_{\geq 0}$ , recursively, by using (7.1), and computing the  $q$ -character of the top module through the  $q$ -characters of other modules in relations in Theorem 7.4.  $\square$*

## 8. DIMENSIONS

In this section, we give dimension formulas for the modules  $\mathcal{B}_{k,\ell}^{(s)}, \mathcal{C}_{k,\ell}^{(s)}, \mathcal{D}_{k,\ell}^{(s)}, \mathcal{E}_{k,\ell}^{(s)}, \mathcal{F}_{k,\ell}^{(s)}, \tilde{\mathcal{B}}_{k,\ell}^{(s)}, \tilde{\mathcal{C}}_{k,\ell}^{(s)}, \tilde{\mathcal{D}}_{k,\ell}^{(s)}, \tilde{\mathcal{E}}_{k,\ell}^{(s)}, \tilde{\mathcal{F}}_{k,\ell}^{(s)}$ .

Note that dimensions do not depend on the upper index  $s$ . Note also that  $\dim M = \dim \tilde{M}$  for each  $M = \mathcal{B}_{k,\ell}^{(s)}, \mathcal{C}_{k,\ell}^{(s)}, \mathcal{D}_{k,\ell}^{(s)}, \mathcal{E}_{k,\ell}^{(s)}, \mathcal{F}_{k,\ell}^{(s)}$ .

**Theorem 8.1.** *Let  $s \in \mathbb{Z}$ ,  $k, \ell \in \mathbb{Z}_{\geq 0}$ . Then*

$$\begin{aligned} \dim \mathcal{B}_{k,3\ell}^{(s)} &= (\ell + 2)(\ell + 1)(1 + k)(k + 3 + \ell)(k + 2 + \ell) \\ &\quad (54\ell^3 k^3 + 243\ell^2 k^3 + 363\ell k^3 + 180k^3 + 2784\ell^2 k^2 + 1080k^2 + 162\ell^4 k^2 \\ &\quad + 2880\ell k^2 + 1134\ell^3 k^2 + 162\ell^5 k + 1539\ell^4 k + 5490\ell^3 k + 9132\ell^2 k + 7057\ell k + \\ &\quad 2040k + 54\ell^6 + 648\ell^5 + 3069\ell^4 + 7272\ell^3 + 8977\ell^2 + 5380\ell + 1200)/14400, \end{aligned}$$

$$\begin{aligned} \dim \mathcal{B}_{k,3\ell+1}^{(s)} &= (\ell + 3)(\ell + 2)(\ell + 1)(1 + k)(k + 2 + \ell)(k + 4 + \ell)(k + 3 + \ell) \\ &\quad (171\ell k^2 + 120k^2 + 54\ell^2 k^2 + 600k + 621\ell^2 k + 108\ell^3 k \\ &\quad + 1116\ell k + 54\ell^4 + 450\ell^3 + 1341\ell^2 + 1665\ell + 700)/14400, \end{aligned}$$

$$\begin{aligned} \dim \mathcal{B}_{k,3\ell+2}^{(s)} &= (\ell + 3)(\ell + 2)(\ell + 1)(1 + k)(k + 4 + \ell)(k + 3 + \ell)(2 + k + \ell) \\ &\quad (300k^2 + 261\ell k^2 + 54\ell^2 k^2 + 891\ell^2 k + 2376\ell k + 2040k \\ &\quad + 108\ell^3 k + 54\ell^4 + 630\ell^3 + 2691\ell^2 + 4995\ell + 3400)/14400, \end{aligned}$$

$$\dim \mathcal{C}_{k,\ell}^{(s)} = (\ell+2)(\ell+1)(k+2)(k+1)(k+3+\ell)(k+2+\ell) \\ (3k^2 + 3\ell k^2 + 12k + 15\ell k + 3\ell^2 k + 3\ell^2 + 12\ell + 10)/240,$$

$$\dim \mathcal{D}_{k,\ell}^{(s)} = (\ell+2)(\ell+1)(k+2)(k+1)(k+3+\ell)(k+4+\ell) \\ (3\ell k^2 + 6k^2 + 3\ell^2 k + 30k + 21\ell k + 6\ell^2 + 30\ell + 35)/240,$$

$$\dim \mathcal{E}_{3k,2\ell}^{(s)} = (\ell+2)(\ell+1)(k+1)(k+\ell+1)(k+\ell+2)^2(k+\ell+3)^2 \\ (27k^4\ell^2 + 81k^4\ell + 54k^4 + 81k^3\ell^3 + 468k^3\ell^2 + 825k^3\ell \\ + 432k^3 + 81k^2\ell^4 + 711k^2\ell^3 + 2184k^2\ell^2 + 2754k^2\ell + 1179k^2 + 27k\ell^5 + \\ 342k\ell^4 + 1593k\ell^3 + 3438k\ell^2 + 3435k\ell + 1260k + 18\ell^5 \\ + 180\ell^4 + 696\ell^3 + 1296\ell^2 + 1160\ell + 400)/28800,$$

$$\dim \mathcal{E}_{3k,2\ell+1}^{(s)} = (\ell+3)(\ell+2)(\ell+1)(k+1)(k+\ell+4)(k+\ell+2)^2(k+\ell+3)^2 \\ (27k^4\ell + 54k^4 + 81k^3\ell^2 + 414k^3\ell + 510k^3 + 81k^2\ell^3 + \\ 684k^2\ell^2 + 1842k^2\ell + 1611k^2 + 27k\ell^4 + 342k\ell^3 + \\ 1512k\ell^2 + 2808k\ell + 1875k + 18\ell^4 + 180\ell^3 + 642\ell^2 + 960\ell + 500)/28800,$$

$$\dim \mathcal{E}_{3k+1,2\ell}^{(s)} = (\ell+2)(\ell+1)(k+1)(k+\ell+4)(k+\ell+2)^2(k+\ell+3)^2 \\ (27k^4\ell^2 + 81k^4\ell + 54k^4 + 81k^3\ell^3 + 477k^3\ell^2 + 852k^3\ell \\ + 450k^3 + 81k^2\ell^4 + 747k^2\ell^3 + 2373k^2\ell^2 + 3069k^2\ell + 1341k^2 + \\ 27k\ell^5 + 387k\ell^4 + 1935k\ell^3 + 4353k\ell^2 + 4461k\ell + 1665k \\ + 36\ell^5 + 360\ell^4 + 1374\ell^3 + 2490\ell^2 + 2140\ell + 700)/28800,$$

$$\dim \mathcal{E}_{3k+1,2\ell+1}^{(s)} = (\ell+3)(\ell+2)(\ell+1)(k+1)(k+\ell+2)(k+\ell+3)^2(k+\ell+4)^2 \\ (27k^4\ell + 54k^4 + 81k^3\ell^2 + 450k^3\ell + 582k^3 + 81k^2\ell^3 + 774k^2\ell^2 + \\ 2310k^2\ell + 2193k^2 + 27k\ell^4 + 414k\ell^3 + 2124k\ell^2 + 4488k\ell + \\ 3375k + 36\ell^4 + 396\ell^3 + 1590\ell^2 + 2760\ell + 1750)/28800,$$

$$\dim \mathcal{E}_{3k+2,2\ell}^{(s)} = (\ell+2)(\ell+1)(k+2)(k+1)(k+\ell+4)(k+\ell+2)(k+\ell+3)^2 \\ (27k^4\ell^2 + 81k^4\ell + 54k^4 + 108k^3\ell^3 + 648k^3\ell^2 + 1176k^3\ell \\ + 630k^3 + 162k^2\ell^4 + 1458k^2\ell^3 + 4629k^2\ell^2 + 6057k^2\ell + \\ 2691k^2 + 108k\ell^5 + 1296k\ell^4 + 5946k\ell^3 + 12942k\ell^2 + 13230k\ell + 4995k \\ + 27\ell^6 + 405\ell^5 + 2439\ell^4 + 7515\ell^3 + 12429\ell^2 + 10395\ell + 3400)/28800,$$

$$\begin{aligned} \dim \mathcal{E}_{3k+2,2\ell+1}^{(s)} &= (\ell+3)(\ell+2)(\ell+1)(k+2)(k+1)(k+\ell+5)(k+\ell+2) \\ &\quad (k+\ell+3)^2(k+\ell+4)^2(9k^2\ell+18k^2+ \\ &\quad 18k\ell^2+99k\ell+128k+9\ell^3+81\ell^2+237\ell+225)/9600, \end{aligned}$$

$$\begin{aligned} \dim \mathcal{F}_{3k,3\ell}^{(s)} &= (\ell+2)^2(\ell+1)^2(k+2)^2(k+1)^2(k+\ell+3)^2 \\ &\quad (27k^4\ell^2+81k^4\ell+54k^4+54k^3\ell^3+405k^3\ell^2+801k^3\ell+432k^3+27k^2\ell^4+ \\ &\quad 405k^2\ell^3+1746k^2\ell^2+2646k^2\ell+1179k^2+81k\ell^4+801k\ell^3+ \\ &\quad 2646k\ell^2+3342k\ell+1260k+54\ell^4+432\ell^3+1179\ell^2+1260\ell+400)/57600, \end{aligned}$$

$$\begin{aligned} \dim \mathcal{F}_{3k+1,3\ell}^{(s)} &= (\ell+2)^2(\ell+1)^2(k+3)(k+1)(k+2)^2(k+\ell+4)(k+\ell+3) \\ &\quad (27k^4\ell^2+81k^4\ell+54k^4+54k^3\ell^3+414k^3\ell^2+828k^3\ell+450k^3+ \\ &\quad 27k^2\ell^4+414k^2\ell^3+1854k^2\ell^2+2907k^2\ell+1341k^2+81k\ell^4+864k\ell^3+ \\ &\quad 3063k\ell^2+4116k\ell+1665k+54\ell^4+498\ell^3+1563\ell^2+1905\ell+700)/57600, \end{aligned}$$

$$\begin{aligned} \dim \mathcal{F}_{3k+2,3\ell}^{(s)} &= (\ell+2)^2(\ell+1)^2(k+3)(k+1)(k+2)^2(k+\ell+4)(k+\ell+3) \\ &\quad (27k^4\ell^2+81k^4\ell+54k^4+54k^3\ell^3+504k^3\ell^2+1098k^3\ell+630k^3+27k^2\ell^4+ \\ &\quad 558k^2\ell^3+3042k^2\ell^2+5355k^2\ell+2691k^2+135k\ell^4+1764k\ell^3+7395k\ell^2+ \\ &\quad 11190k\ell+4995k+162\ell^4+1734\ell^3+6249\ell^2+8475\ell+3400)/57600, \end{aligned}$$

$$\begin{aligned} \dim \mathcal{F}_{3k,3\ell+1}^{(s)} &= (\ell+3)(\ell+1)(\ell+2)^2(k+2)^2(k+1)^2(k+\ell+4)(k+\ell+3) \\ &\quad (27k^4\ell^2+81k^4\ell+54k^4+54k^3\ell^3+414k^3\ell^2+864k^3\ell+498k^3+ \\ &\quad 27k^2\ell^4+414k^2\ell^3+1854k^2\ell^2+3063k^2\ell+1563k^2+81k\ell^4+828k\ell^3+ \\ &\quad 2907k\ell^2+4116k\ell+1905k+54\ell^4+450\ell^3+1341\ell^2+1665\ell+700)/57600, \end{aligned}$$

$$\begin{aligned} \dim \mathcal{F}_{3k+1,3\ell+1}^{(s)} &= (\ell+3)(\ell+1)(\ell+2)^2(k+3)(k+1)(k+2)^2(k+\ell+3)(k+\ell+4) \\ &\quad (27k^4\ell^2+81k^4\ell+54k^4+54k^3\ell^3+450k^3\ell^2+972k^3\ell+570k^3+27k^2\ell^4+ \\ &\quad 450k^2\ell^3+2214k^2\ell^2+3891k^2\ell+2061k^2+81k\ell^4+972k\ell^3+3891k\ell^2+ \\ &\quad 6060k\ell+2985k+54\ell^4+570\ell^3+2061\ell^2+2985\ell+1400)/57600, \end{aligned}$$

$$\begin{aligned} \dim \mathcal{F}_{3k+2,3\ell+1}^{(s)} &= (\ell+3)(\ell+1)(\ell+2)^2(k+3)(k+1)(k+2)^2(k+\ell+3)(k+\ell+5) \\ &\quad (k+\ell+4)^2(9k^2\ell^2+27k^2\ell+18k^2+45k\ell^2+135k\ell \\ &\quad +88k+54\ell^2+164\ell+105)/19200, \end{aligned}$$

$$\begin{aligned} \dim \mathcal{F}_{3k, 3\ell+2}^{(s)} &= (\ell+3)(\ell+1)(\ell+2)^2(k+2)^2(k+1)^2(k+\ell+4)(k+\ell+3) \\ &\quad (27k^4\ell^2 + 135k^4\ell + 162k^4 + 54k^3\ell^3 + 558k^3\ell^2 + 1764k^3\ell + 1734k^3 + 27k^2\ell^4 \\ &\quad + 504k^2\ell^3 + 3042k^2\ell^2 + 7395k^2\ell + 6249k^2 + 81k\ell^4 + 1098k\ell^3 + 5355k\ell^2 + \\ &\quad 11190k\ell + 8475k + 54\ell^4 + 630\ell^3 + 2691\ell^2 + 4995\ell + 3400)/57600, \end{aligned}$$

$$\begin{aligned} \dim \mathcal{F}_{3k+1, 3\ell+2}^{(s)} &= (\ell+3)(\ell+1)(\ell+2)^2(k+3)(k+1)(k+2)^2(k+\ell+3)(k+\ell+5) \\ &\quad (k+\ell+4)^2(9k^2\ell^2 + 45k^2\ell + 54k^2 + 27k\ell^2 + 135k\ell \\ &\quad + 164k + 18\ell^2 + 88\ell + 105)/19200, \end{aligned}$$

$$\begin{aligned} \dim \mathcal{F}_{3k+2, 3\ell+2}^{(s)} &= (\ell+3)(\ell+1)(\ell+2)^2(k+3)(k+1)(k+2)^2(k+\ell+4)(k+\ell+5) \\ &\quad (27k^4\ell^2 + 135k^4\ell + 162k^4 + 54k^3\ell^3 + 630k^3\ell^2 + 2124k^3\ell + 2166k^3 + 27k^2\ell^4 + \\ &\quad 630k^2\ell^3 + 4374k^2\ell^2 + 11661k^2\ell + 10473k^2 + 135k\ell^4 + 2124k\ell^3 + 11661k\ell^2 + \\ &\quad 26748k\ell + 21759k + 162\ell^4 + 2166\ell^3 + 10473\ell^2 + 21759\ell + 16400)/57600. \end{aligned}$$

*Proof.* We check the initial conditions, namely dimensions of  $\mathcal{B}_{0,1}^{(s)}, \mathcal{B}_{1,0}^{(s)}$ . We check the dimensions are compatible with relations (3.1), (3.2), (3.3). We directly check that the formulas satisfy the relations in Theorems 3.4. For the checks we employed the computer algebra system Maple.

The theorem follows since the solution of the extended  $T$ -system is unique, see Proposition 3.6.  $\square$

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